RELATIONS BETWEEN HYPERGROUPS OF LINEAR DIFFERENTIAL OPERATORS IN THE JACOBI FORM

Štěpán Křehlík
Doctoral Degree Programme (3), FEEC BUT
E-mail: xkrehl02@stud.feec.vutbr.cz

Supervised by: Jan Chvalina
E-mail: chvalina@feec.vutbr.cz

Abstract: In the contribution there are studied relations on the space of hyperstructures. We describe basic and compatibility types of binary relations on hypergroups determined by linear differential operators of the second order in the Jacobi form. These hyperstuctures are motivated by modelling time functions.

Keywords: Relations, hypergroup, differential operator, modelling function

1 INTRODUCTION

In my research I study hyperstructures and multiautomata which are based on linear differential operators constructed from various types of signals. Some results of this research have been included in [4, 5, 8, 9]. The research is based on results achieved by Chvalina [1, 2, 3] and [10] and concepts, notation and terminology of hyperstructure theory. For general concepts c.f. [6, 7]; some results are based on [11, 12]

The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup. Recall some basic notions and notation of the hypergroup theory from [1, 2, 5, 7, 8, 11, 12]. *A hypergroupoid* is a pair $(H,\bullet)$, where $H \neq 0$ and $\bullet : H\times H \longrightarrow \mathcal{P}^*(H)$ is binary hyperoperation on H (here $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H). If $a\bullet(b\bullet c) = (a\bullet b)\bullet c$ holds for all $a,b,c \in H$, then $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom ($a\bullet H = H = H\bullet a$ for any element $a \in H$) is satisfied, then the semihypergroup $(H,\bullet)$ is called a *hypergroup*. For two arbitrary non–empty subsets A,B of the set H we define

$$A\bullet B = \bigcup\{a\bullet b; a \in A, b \in B\}.$$

A hypergroup $(H,\bullet)$ is called a transposition hypergroup (or a join space), if the hypergroup satisfies the transposition axiom: For any quadruple $a,b,c,d \in H$ the relationship $b\setminus a \approx c/d$ implies $a\bullet d \approx b\bullet c$, where sets $b\setminus a = \{x \in H; a \in b\bullet x\}$, $c/d = \{x \in H; c \in x\bullet d\}$ are called left and right extensions, respectively.

2 HYPERGROUPS OF LINEAR DIFFERENTIAL OPERATORS IN THE JACOBI FORM

First, we consider the function of the Gaussian–shaped pulse signal $v(t) = a\cdot\exp(-2\pi t^2)$, and its first and second derivatives $v'(t) = -4a\pi t\exp(-2\pi t^2)$, $v''(t) = 16a\pi^2t^2v(t)$ for any $t \in$

$\langle 0,\infty)$. Then we have differential equation

$$v''(t) - 16a\pi t^2 v(t) = 0,\ \ t \in \langle 0,\infty)$$

with initial conditions $v(0) = a, v'(0) = 0$; for more information see [4, 9, 10]. In what follows we will consider linear differential operators in the so called Jacobi form. O. Borůvka investigated the second order linear differential equations in the Jacobi form, i.e. $y'' + p(x)y = 0$, where p is continuous function. With respect to importance of equations in the Jacobi form we will investigated hypergroups of operators $L(0,p)$ where $L(0,p) = y'' + p(x)y$. So we will consider linear differential operators of the so called Jacobi form $L(0,\Psi(a)t) = \frac{d^2}{dt^2} + 16a\pi t^2 Id$, where $a \in \mathbb{R}^+$. This differential operator is formed by the left hand side of the above differential equation. The set $\mathbb{JA}_2(T)$ is a set of differential operators in the Jacobi form $L(0,\Psi(a)t)$, which are motivated by functions. We define hyperoperation "$*$" on this set by the rule: For $L(0,\Psi(a)t), L(0,\Psi(b)t) \in \mathbb{JA}_2(T)$, we put

$$L(0,\Psi(a)t) * L(0,\Psi(b)t) \ = \ \{L(0,\Psi(c)t); c \in \mathbb{R}^+, a \cdot b \leq c,\ t \in \langle 0,\infty)\}.$$

Then $(\mathbb{JA}_2(T),*)$ is a commutative hypergroup satisfying the transposition axiom, i.e. it is a join space; for the proof see [9].

However, let us deal with another (yet unpublished) case. Chapman-Richardson's function (CHRF) $y = A \cdot [1 - exp(-ct)]^b$ and its first and second derivation $y' = Abc[1 - exp(-ct)]^{b-1} \cdot exp(-ct),\ \ y'' = -Abc^2 \cdot [1 - exp(-ct)]^b \cdot \frac{exp(ct)-b}{(exp(ct)-1)^2}$, thus

$$y'' + bc^2 \cdot \frac{exp(ct) - b}{(exp(ct) - 1)^2} y = 0; \quad t \in \langle 0,\infty)$$

Modelling function $y = A \cdot [1 - exp(-ct)]^b$ is one of the most common functions based on the original Bertalanffy equation derived for growth and increment of body weight. As above, we define the set $\mathbb{J}_g\mathbb{A}_2(T) = \{L(0,\tau(b,c)t); b,c \in \mathbb{R}^+\}$, where $L(0,\tau(b,c)t)$ is a linear second order differential operator and $L(0,\tau(b,c)t)y = 0$, thus $\tau(b,c)t = bc^2 \cdot \frac{exp(ct)-b}{(exp(ct)-1)^2}$. We define a hyperoperation "#" by the rule:

$$L(0,\tau(b,c)t) \# L(0,\tau(d,e)t) = \{L(0,\tau(k,l)t); k,l \in \mathbb{R}^+, k \geq b \cdot d, l \geq c \cdot e\}$$

for any $L(0,\tau(b,c)t), L(0,\tau(d,e)t) \in \mathbb{J}_g\mathbb{A}_2(T)$.

Lemma 2.1 *Let $\{L(0,\tau(a_i,b_i)t); a_i,b_i \in \mathbb{R}^+, i = 1,2,3,4\}$ be a four-element subset of the hypergroup $\mathbb{J}_g\mathbb{A}_2(T)$. Then we have*

$$L(0,\tau(a_1,b_1)t) \# L(0,\tau(a_2,b_2)t) \approx L(0,\tau(a_3,b_3)t) \# L(0,\tau(a_4,b_4)t).$$

Proof. Suppose $L(0,\tau(a_i,b_i)t), i = 1,2,3,4$ are arbitrary operators from the set $\mathbb{J}_g\mathbb{A}_2(T)$. Denote $c = max\{a_1a_2, a_3a_4\}, d = max\{b_1b_2, b_3b_4\}$. Since $a_k, b_k \in \mathbb{R}^+$, we have $a_1a_2 \leq c, a_3a_4 \leq c, b_1b_2 \leq d, b_3b_4 \leq d$, thus $L(0,\tau(c,d)t) \in L(0,\tau(a_1,b_1)t) \# L(0,\tau(a_2,b_2)t) \cap L(0,\tau(a_3,b_3)t) \# L(0,\tau(a_4,b_4)t)$, consequently the assertion of the lemma is valid. $\square$

Theorem 2.2 *Let $T = \langle 0, \infty)$ be the interval of real non-negative numbers. Then the commutative hypergroupoid $(\mathbb{J}_g\mathbb{A}_2(T), \#)$ with the above defined hyperoperation $\#$ is a commutative hypergroup satisfying the transposition axiom, i.e. it is a commutative transposition hypergroup hence a join space.*

Proof. For arbitrary operators $L(0,\tau(a,b)t), L(0,\tau(c,d)t) \in \mathbb{J}_g\mathbb{A}_2(T)$ we have $L(0,\tau(a,b)t)\#L(0,\tau(c,d)t) = \{L(0,\tau(r,s)t); r,s \in \mathbb{R}^+, r \geq ac, s \geq bd\}$. Clearly, the hypergroupoid $(\mathbb{J}_g\mathbb{A}_2(T), \#)$ is commutative. We are going to show the hyperoperation is associative: Suppose $L(0,\tau(a,b)t), L(0,\tau(c,d)t), L(0,\tau(e,f)t) \in \mathbb{J}_g\mathbb{A}_2(T)$. Then $L(0,\tau(a,b)t)\#(L(0,\tau(c,d)t)\#L(0,\tau(e,f)t)) = L(0,\tau(a,b)t)\#\{L(0,\tau(k,l)t), k,l \in \mathbb{R}^+, k \geq ce, l \geq df\}$. We have $L(\tau) = \{L(0,\tau(k,l)t), k,l \in \mathbb{R}^+, k \geq ce, l \geq df\}$ then $\bigcup\{L(0,\tau(a,b)t)\#L(0,\tau(m,n)t);$ $L(0,\tau(m,n)t) \in L(\tau)\} = \{L(0,\tau(r,s)t); r,s \in \mathbb{R}^+, r \geq ace, s \geq bdf\} = \{L(0,\tau(k,l)t); k,l \in \mathbb{R}^+,$ $k \geq ac, l \geq bd\}\#L(0,\tau(e,f)t) = (L(0,\tau(a,b)t)\#L(0,\tau(c,d)t))\#L(0,\tau(e,f)t)$. It holds that the hypergroupoid $(\mathbb{J}_g\mathbb{A}_2(T), \#)$ is a commutative semihypergroup.
Further, $L(0,\tau(a,b)t)\#\mathbb{J}_g\mathbb{A}_2(T) = \mathbb{J}_g\mathbb{A}_2(T)$. Evidently $L(0,\tau(a,b)t)\#\mathbb{J}_g\mathbb{A}_2(T) \subseteq \mathbb{J}_g\mathbb{A}_2(T)$. Consider an arbitrary operator $L(0,\tau(m,n)t) \in \mathbb{J}_g\mathbb{A}_2(T)$. Then there exists an operator $L(0,\tau(r,s)t) \in \mathbb{J}_g\mathbb{A}_2(T)$ such that

$$L(0,\tau(m,n)t) \in L(0,\tau(a,b)t)\#L(0,\tau(r,s)t), i.e.\ a \cdot r \leq m,\ b \cdot s \leq n.$$

Indeed, define coefficients $r = \frac{m}{a}$ and $s = \frac{n}{b}$. Then $a \cdot \frac{m}{a} = m$ and $b \cdot \frac{n}{b} = n$, thus $L(0,\tau(m,n)t) \in L(0,\tau(a,b)t)\#L(0,\tau(r,s)t)$, where $L(0,\tau(s,r)t) \in \mathbb{J}_g\mathbb{A}_2(T)$. Consequently $\mathbb{J}_g\mathbb{A}_2(T) \subseteq L(0,\tau(a,b)t)\#\mathbb{J}_g\mathbb{A}_2(T)$ hence we obtain

$$L(0,\tau(a,b)t)\#\mathbb{J}_g\mathbb{A}_2(T) = \mathbb{J}_g\mathbb{A}_2(T).$$

We are going to show that the hypergroup is a join space. The hyperoperation "#" is commutative thus $L(0,\tau(a,b)t) \setminus L(0,\tau(c,d)t) = L(0,\tau(c,b)t)/L(0,\tau(a,b)t)$ for all operators $L(0,\tau(a,b)t), L(0,\tau(c,d)t) \in \mathbb{J}_g\mathbb{A}_2(T)$.

It remains to prove that this hypergroup satisfies the transposition law.
Suppose $L(0,\tau(a,b)t), L(0,\tau(c,d)t), L(0,\tau(e,f)t), L(0,\tau(g,h)t) \in \mathbb{J}_g\mathbb{A}_2(T)$ is a quadruple of differential operators in the Jacobi form such that

$$L(0,\tau(a,b)t)/L(0,\tau(c,d)t) \approx L(0,\tau(e,f)t)/L(0,\tau(g,h)t).$$

Denote $u = max\{ag, ce\}, v = max\{bh, df\}$. Since $u \geq ag, v \geq bh$, we have $L(0,\tau(u,v)t) \in L(0,\tau(a,b)t)\#L(0,\tau(g,h)t)$. Similarly $L(0,\tau(u,v)t) \in L(0,\tau(c,d)t)\#L(0,\tau(e,f)t)$ which is a consequence of inequalities $u \geq ce, v \geq df$.

Thus

$$L(0,\tau(a,b)t)\#L(0,\tau(g,h)t) \cap L(0,\tau(c,d)t)\#L(0,\tau(e,f)t) = \emptyset$$

Consequently the commutative hypergroup $(\mathbb{J}_g\mathbb{A}_2(T), \#)$ satisfies the transposition law therefore it is a transposition hypergroup, i.e. a join space. $\square$

3 PROPERTIES OF RELATIONS BETWEEN HYPERGROUPS

Define a binary relation $\rho \subseteq \mathbb{J}_g\mathbb{A}_2(T) \times \mathbb{J}\mathbb{A}_2(T)$ by the rule $[L(0,\tau(b,c)t), L(0,\Psi(a)t)] \in \rho$ for an arbitrary pair $[L(0,\tau(b,c)t), L(0,\Psi(a)t)] \in \mathbb{J}_g\mathbb{A}_2(T) \times \mathbb{J}\mathbb{A}_2(T)$, whenever there exists $L(0,\tau(d,e)t) \in \mathbb{J}_g\mathbb{A}_2(T)$ such that

$$a = b \cdot d + c \cdot e \qquad \text{for the coeficients } a,b,c,d,e \in \mathbb{R}^+.$$

In [3] some compatibility properties of binary relations on hypergroups are treated in detail. All of them can be extended onto cases of relations between different structures.

So, let G,H be hypergroups and $R \subseteq H \times G$. We say that the binary relation R has the transmission substitution property (TSP):

- TSP-1; of the first type
  if for any pair $[a,b] \in H \times G, [c,d] \in H \times G$ such that $[a,b] \in R, [c,d] \in R$ we have $(a \cdot c)\overline{R}(b \cdot d)$, i.e. for each $x \in a \cdot c$ there is $y \in b \cdot d$ such that xRy and vice versa, for any $y \in b \cdot d$ there exists an element $x \in a \cdot c$ with xRy.
- TSP-2; of the second type
  if for any pair $[a,b] \in H \times G, [c,d] \in H \times G$ such that $[a,b] \in R, [c,d] \in R$ we have $R(a \cdot c) = b \cdot d$
- TSP-3; of the third type
  if for any pair $[a,b] \in H \times G, [c,d] \in H \times G$ such that $[a,c] \in R, [b,d] \in R$ implies $(a \cdot c)\overline{\overline{R}}(b \cdot d)$, i.e. for any pair $[x,y] \in (a \cdot c) \times (b \cdot d)$ we have xRy, i.e. $(a \cdot c) \times (b \cdot d) \subseteq R$.

Theorem 3.1 *Let $\rho \subseteq \mathbb{J}_g\mathbb{A}_2(T) \times \mathbb{J}\mathbb{A}_2(T)$ be the relation defined as above. Then the relation ρ has transmission substitution properties TSP-1 and TSP-3.*

Proof. Consider arbitrary pairs of operators $[L(0,\tau(b,c)t), L(0,\Psi(a)t)] \in \mathbb{J}_g\mathbb{A}_2(T) \times \mathbb{J}\mathbb{A}_2(T)$, $[L(0,\tau(e,f)t), L(0,\Psi(d)t)] \in \mathbb{J}_g\mathbb{A}_2(T) \times \mathbb{J}\mathbb{A}_2(T)$ such that pairs $[L(0,\tau(b,c)t),\, L(0,\Psi(a)t)] \in \rho$ and $[L(0,\tau(e,f)t),\, L(0,\Psi(d)t)] \in \rho$.

TSP-1:
Consider the hyperproduct $L(0,\Psi(a)t) * L(0,\Psi(d)t) = \{L(0,\Psi(g)t), g \in \mathbb{R}^+, g \geq a \cdot d\}$ and hyperproduct $L(0,\tau(b,c)t)\#L(0,\tau(e,f)t) = \{L(0,\tau(u,v)t); u,v \in \mathbb{R}^+, u \geq b \cdot e, v \geq c\dot{f}\}$. Suppose $L(0,\tau(m,n)t) \in \{L(0,\tau(u,v)t); u,v \in \mathbb{R}^+, u \geq b \cdot e, v \geq c\dot{f}\}$ and $r \in \mathbb{R}^+$ such that $r \geq \frac{a \cdot d}{m+n}$. Then $mr + nr = (m+n)r \geq ad$. Denoting $(m+n)r = s$ we have $L(0,\Psi(s)t) \in L(0,\Psi(a)t) * L(0,\Psi(d)t)$ and simultaneously $L(0,\tau(m,n)t)\,\rho L(0,\Psi(s)t)$ and vice versa in the same way.

TSP-3:
Consider that for any pair $[L(0,\tau(u,v)t), L(0,\Psi(r)t)] \in \{L(0,\tau(u,v)t), u,v \in \mathbb{R}^+, u \geq be, v \geq cf\} \times \{L(0,\Psi(r)t); t \in \mathbb{R}^+, r \geq ad\}$ there exists an operator $L(0,\tau(s,s)t) \in L(0,\tau(b,c)t)\#$ $L(0,\tau(e,f)t)$ (here $s = \frac{s}{u+v} \geq 0$) such that $us + vs = (u+v)s = r$. Then pars $[L(0,\tau(u,v)t),$ $L(0,\Psi(r)t)] \in \rho$.

Remark 3.2 *It is easy to see that the relation ρ does not have the properties TSP-2.*
Indeed, consider $\rho(L(0,\tau(b,c)t)\#L(0,\tau(e,f)t)) = \rho(\{L(0,\tau(u,v)t); u \geq be, v \geq ef\})$. Suppose

$a=10, d=50, b=2, e=1, c=3, f=2$. Then $L(0,\Psi(a)t)*L(0,\Psi(d)t)=\{L(0,\Psi(s)t); s\geq ad\}$ i.e. if $s\geq 5\cdot 10^2$. Next for $m=0,1=n$ we have $L(0,\tau(2,6)t)\in L(0,\tau(2,3)t)\#L(0,\tau(1,2)t)$ and for $a=2\cdot 0,1+6\cdot 0,1=0,8$ the operator $L\left(0,\Psi(8\cdot 10^{-1})t\right)\in\rho(L(0,\tau(2,6)t))$. Since $8\cdot 10^{-1}, L\left(0,\Psi(8\cdot 10^{-1})t\right)\notin L(0,\Psi(10)t)*L(0,\Psi(50)t)$. Thus $\rho(L(0,\tau(2,3)t)\#L(0,\tau(1,2)t))$ $\neq L(0,\Psi(10)t)*L(0,\Psi(50)t)$. □

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