QUASI–MULTIAUTOMATA FORMED BY CONTINUOUS FUNCTIONS AND BY SECOND-ORDER LINEAR DIFFERENTIAL OPERATORS IN THE JACOBI FORM

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ABSTRACT

We investigate quasi–multiautomata in which as input–sets are used (semi–)rings of smooth continuous positive functions and state–sets are formed by linear differential operators in the Jacobi form. Further we investigate certain properties of such structures and the same properties are examined for quasi–multiautomata input–sets and state–sets are interchanged.

1 INTRODUCTION

The contribution is devoted to quasi–multiautomata equipped by state (semi–) hypergroup of second-order linear differential operators in the Jacobi form. The quasi–multiautomata are one of the basic theoretical resources for modelling of discrete computing systems. Modifications of these concepts are described and studied in many papers, these concepts are further perceived as concrete specifications of general input-output systems . Founder of the general system theory was Austrian Karel Ludwig von Bertalanffy. His mathematical model of an origanism's growth over time, published in 1934, is still in used today. This contribution is a continuation of [7] where some motivation factors are mentioned. Cf. also [4] Constructions of algebraic binary hyperstructures (semihypergroups and hypergroups) from ordered algebraic systems are based on a certain lemma on principal ends generated by products of pairs of elements know as Ends-lemma. cf. [2, 3, 7, 9, 10]. In the present contribution we construct actions of commutative transposition hypergroups i.e. join spaces created from rings of continuous and smooth functions of a given class on semihypergroups or hypergroups of second order linear ordinary differential operators.

2 HYPERGROUPS AND AUTOMATA

Let J be an open interval of real numbers, $\mathbb{C}(J)$ be the ring of all continuous functions on J and $\mathbb{C}^+(J)$ its subsemirings of all positive functions. In what follows we denote $L(p,q)y = y'' + p(x)y' + q(x)y$ and $L(0,q)y = y'' + q(x)y; p,q \in \mathbb{C}(J)$. Otakar Borůvka has obtained a criterion of a global equivalence for second order differential equations within the Jacobi form, i.e.

$$y'' + q(x) \cdot y = 0,\ q \in \mathbb{C}(J).$$

and he also found corresponding global canonical forms for such equations. This is motivation of the investigation of operators in question. The other is quite concrete: Considering a two–parameter model $y(t) = t^2 e^{-\lambda t}$ of certain non periodic signals, we obtain that these functions satisfy differential equation $y''(t) + q(t) \cdot y(t) = 0$ where $q(t) = (4\lambda t - \lambda^2 t^2 - 2) \cdot t^{-2}$ We will consider the bellow defined operation on the set of such differential operators (under the supposition $q(x) \neq 0,\ x \in J$). Denote by $\mathbb{JA}_2(J)$ the set of operators defined

$$\mathbb{JA}_2(J) = \{L(0,q);\ q \in \mathbb{C}^+(J)\}$$

and its subset

$$\mathbb{J}_C\mathbb{A}_2(J) = \{L(0,r);\ r \in \mathbb{R}^+\}.$$

Recall some basic notions and notation of the hypergroup theory from c.f. [1, 3, 4, 7]. *A hypergroupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \longrightarrow \mathcal{P}^*(H)$ is a binary hyperoperation on H. (Here $\mathcal{P}^*(H)$ denotes the system of all nonempty subsets of H). If $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$ then $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom ($a \bullet H = H = H \bullet a$ for any element $a \in H$) is satisfied, then the pair $(H, \bullet)$ is called a *hypergroup*.

The set $\mathbb{JA}_2(J)$ is a set of differential operators in the Jacobi form. On this set we define hyperoperation "$*$" by the rule: for $L(0,p),\ L(0,q) \in \mathbb{JA}_2(J)$, we put $L(0,p) * L(0,q) = \{L(0,\varphi); p(x) \cdot q(x) \leq \varphi(x);\ \varphi \in \mathbb{C}_+(J)\}$. Then $(\mathbb{JA}_2(J), *)$ is a commutative hypergroup; for the prof see [7]. Its subset $\mathbb{J}_C\mathbb{A}_2(J)$ is a set of differential operators in the Jacobi form with constant coefficients. On this set we define hyperopreation "$*_C$" by the rule: For $L(0,s),\ L(0,r) \in \mathbb{J}_C\mathbb{A}_2(J)$, we put $L(0,s) *_C L(0,r) = \{L(0,t); s \cdot r \leq t;\ t \in \mathbb{R}^+\}$. This hypergrupoid $(\mathbb{J}_C\mathbb{A}_2(J), *_C)$ is a commutative subhypergroup of the hypergroup $(\mathbb{JA}_2(J), *)$: For the proof see [7].

These hypergroups will create quasi–multiautomata (without output function) in the role of input alphabets. We are going to observe some their specific properties.

Definition 2.1 *[1] Automaton without output is a triad $\mathbb{A} = (A, S, \delta)$ where A, S are non-empty sets and: $\delta :\ A \times S \to S$ is a transition map satisfying these conditions:*

1) $\delta(e,s) = s$ for any $s \in S$ and the identity $e \in A$, if it exists (the identity condition)

2) $\delta(b, \delta(a,\ s)) = \delta(a \cdot b,\ s)$ for all $a,\ b\ \in A,\ s \in S$ (the mixed associativity condition).

Set S is called the state–set of the automaton $\mathbb{A}$, the set A is called the set of input symbols of the automaton $\mathbb{A}$ and δ is called the transition functions. Elements of the set S are called states, elements of the set A are called words.

Definition 2.2 *[1] Quasi–multiautomaton without outputs is a triad $\mathbb{M} = (H, S, \delta)$ where $(H, \cdot)$ is a semi–hypergroup, S is a non–empty set and: $\delta :\ A \times S \to S$ is a transition map satisfying this condition:*

1) $\delta(b, \delta(a,\ s)) \in \delta(a \cdot b,\ s)$ for all $a,\ b\ \in A,\ s \in S$ (the generalized mixed associativity condition - GMAC).

Set S is called the state–set of the quasi–multiautomaton $\mathbb{M}$, the structure $(H, \cdot)$ is called the input (semi–)hypergroup of the quasi–multiautomaton $\mathbb{M}$ and δ is called the transition functions. Elements of the set S are called state, elements of the set A are called input symbols (or words).

3 PROPERTIES OF STATE AUTOMATA

We consider a structure $\mathbb{A}_1 = ((\mathbb{C}_+(J), \oplus), \mathbb{JA}_2(J), \delta_1)$ where $f \oplus g = (\bigcup_{[a,b]\in \mathbb{R}_0^+ \times \mathbb{R}_0^+} [af + bg)_{\leq}$ By theorem 3.2 [5] the hypergroupoid $(\mathbb{C}_+(J), \oplus)$ is a hypergroup satisfying transposition axiom, thus it is a join space [2], and the set $\mathbb{JA}_2(J)$ is a state set. Define $\delta : (\mathbb{C}_+(J), \oplus) \times \mathbb{JA}_2(J) \longrightarrow \mathbb{JA}_2(J)$ by

$$\delta_1(f, L(0,q)) = L(0, f + q)$$

This structure is satisfying GMAC

$$\delta_1(g, \delta_1(f, L(0,q))) \in \delta_1(f \oplus g, L(0,q)),$$

which is verified in paper [7].

Further we consider a structure $\mathbb{A}_2 = ((\mathbb{C}_+(J), \odot), \mathbb{JA}_2(J), \delta_2)$, where $f \odot g = (\bigcup_{\alpha \in \mathbb{R}_0^+} [\alpha f \cdot g)_{\leq}$ The hypergroupoid $(\mathbb{C}_+(J), \odot)$ is a hypergroup satisfying transposition axiom, thus it is a join space [2, 5] Define $\delta_2 : (\mathbb{C}_+(J), \odot) \times \mathbb{JA}_2(J) \longrightarrow \mathbb{JA}_2(J)$ by

$$\delta_2(f, L(0,q)) = L(0, f \cdot q)$$

This structure is satisfying GMAC

$$\delta_2(f, \delta_2(g, L(0,q))) \in \delta_2(f \odot g, L(0,q))$$

Indeed, suppose $L(0,q) \in \mathbb{JA}_2(J)$ and $f, g \in \mathbb{C}_+(J)$.
The left hand side:

$$\delta_2(f, \delta_2(g, L(0,q))) = \delta_2(f, (L(0, g \cdot q))) = L(0, f \cdot g \cdot q)$$

The right hand side is of the form:

$$\delta_2(f \odot g, L(0,q)) = \delta_2(\bigcup_{\alpha \in \mathbb{R}^+} [\alpha f \cdot g)_{\leq},\ L(0,q) = \{L(0, q \cdot \varphi); \exists \alpha \in \mathbb{R}^+ : \alpha f \cdot g \leq \varphi\}.$$

For $\alpha = 1$ and $\varphi = f \cdot g$ we have $L(0, q \cdot \varphi) \in \delta_2(f \odot g, L(0,q))$

This structure $\mathbb{A}_2$ is satisfying GMAG thus it is quasi–multiautomaton.

Definition 3.1 *[1] A quasi–multiautomaton $\mathbb{A} = (H, S, \delta)$ is said to be:*

- ***Abelian** (or comutative) if $\delta(s, x \cdot y) = \delta(s, y \cdot x)$ for any triad $[s, x, y] \in S \times H \times H$,*
- ***Cyclic** if there is a state $s \in S$ such that for any state $t \in S$ there exists an element $a \in H$ with the property $\delta(s, a) = t$.*

Remark 3.2 *It is not difficult to show the quasi–multiautomaton $\mathbb{A}_1$ is not cyclic. Indeed, suppose $L(0,q), L(0,p) \in \mathbb{JA}_2(J)$ and $f \in \mathbb{C}_+(J)$, then $\delta_1(L(0,q), f) = L(0, q + f)$ for $q = p - f$ we have $\delta_1(L(0,q), f) = L(0, p - f + f) = L(0,p)$. However, function $q = p - f \notin \mathbb{C}_+(J)$, for $p(x) = 2x$ and $f(x) = x^2$ then $q(x) = -x^2 + 2x$ thus $q \notin \mathbb{C}_+(J)$, whenever $(-\infty, 0) \cap J$ or $(0, \infty) \cap J$ are non–empty intervals.*

Proposition 3.3 *Quasi–multiautomata $\mathbb{A}_1, \mathbb{A}_2$ are abelian and the quasi–multiautomaton $\mathbb{A}_2$ is cyclic.*

Proof. Suppose $L(0,q) \in \mathbb{J}\mathbb{A}_2(J)$ and $f, g \in \mathbb{C}_+(J)$.
Then $f \oplus g = (\bigcup_{[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+} [af + bg)_{\leq}$ and $\delta_1 : (\mathbb{C}_+(J), \oplus) \times \mathbb{J}\mathbb{A}_2(J) \longrightarrow \mathbb{J}\mathbb{A}_2(J)$, is defined by $\delta_1(f, L(0,q)) = L(0, f+q)$
Evidently the hyperoperation "$\oplus$" on $\mathbb{C}_+(J)$ is commutative. Thus

$$\delta_1(L(0,q), f \oplus g) = \delta_1(L(0,q), g \oplus f)$$

for any triad $[L(0,q), f, g] \in \mathbb{J}\mathbb{A}_2(J) \times \mathbb{C}_+(J) \times \mathbb{C}_+(J)$.
So the quasi–multiautomaton $\mathbb{A}_1 = ((\mathbb{C}_+(J), \oplus), \mathbb{J}\mathbb{A}_2(J), \delta_1)$ is *abelian (or commutative).* And in the same way we can show that hyperoperation "$\odot$" is commutative. Thus

$$\delta_2(L(0,q), f \odot g) = \delta_2(L(0,q), g \odot f)$$

and the structure $\mathbb{A}_2 = ((\mathbb{C}_+(J), \odot), \mathbb{J}\mathbb{A}_2(J), \delta_2)$ is also *abelian (or commutative).*

We show that the structure $\mathbb{A}_2 = ((\mathbb{C}_+(J), \odot), \mathbb{J}\mathbb{A}_2(J), \delta_2)$ is cyclic. Suppose $L(0,q), L(0,p) \in \mathbb{J}\mathbb{A}_2(J), f \in \mathbb{C}_+(J)$ and $\delta_2 : (\mathbb{C}_+(J), \odot) \times \mathbb{J}\mathbb{A}_2(J) \longrightarrow \mathbb{J}\mathbb{A}_2(J)$, is defined by $\delta_2(f, L(0,q)) = L(0, f \cdot q)$.
Then $\delta_2(L(0,q), f) = L(0, q \cdot f)$; for $q = \frac{p}{f}$ thus we have $\delta_2(L(0,q), f) = L(0, q \cdot f) = L(0, \frac{p}{f} \cdot f) = L(0,p)$, i.e. the quasi–multiautomaton $\mathbb{A}_2$ is cyclic.

From the above consideration there follows that $\mathbb{A}_2$ is strongly connected which means that for any pair of states s_1, s_2 there exist an input symbol $a \in A$ with $\delta_2(a, s_1) = s_2$.

Remark 3.4 *We can obtaine the structure $\mathbb{A}_J = ((\mathbb{J}\mathbb{A}_2(J), *), \mathbb{C}_+(J), \delta_J)$, where $(\mathbb{J}\mathbb{A}_2(J), *)$ is the (semi–)hypergroup of inputs, $\mathbb{C}_+(J)$ is the state–set and $\delta : (\mathbb{J}\mathbb{A}_2(J), \star) \times \mathbb{C}_+(J) \longrightarrow \mathbb{C}_+(J)$ is defined by:*

$$\delta_J(L(0,q), f) = q \cdot f.$$

This structure is satisfying GMAC:

$$\delta_J(L(0,q), \delta_J(L(0,p), f)) \in \delta_J(L(0,q) * L(0,p), f)$$

Indeed, suppose $L(0,q), L(0,p) \in \mathbb{J}\mathbb{A}_2(J)$ and $f \in \mathbb{C}_+(J)$.
The left hand side: $\delta_J(L(0,q), \delta_J(L(0,p), f)) = \delta_J(L(0,q), p \cdot f) = p \cdot q \cdot f$

The right hand side is of the form: $\delta_J(L(0,q) * L(0,p), f) = \{\delta_J(L(0,\varphi), f); p \cdot q \leqq \varphi\} = \{\varphi \cdot f; p \cdot q \leqq \varphi, \varphi \in \mathbb{C}_+(J)\}$.Putting $p \cdot q = \varphi$ we obtain $\delta_J(L(0,q), \delta_J(L(0,p), f)) = p \cdot q \cdot f \in \{\varphi \cdot f; p \cdot q \leqq \varphi, \varphi \in \mathbb{C}_+(J)\} = \delta_J(L(0,p) * L(0,q), f)$ for arbitrary function $f \in \mathbb{C}_+(J)$.

The structure $\mathbb{A}_J$ is satisfying GMAG and thus it is a quasi–multiautomaton.

The quasi–multiautomaton $\mathbb{A}_J$ has the same properties as the above structure $\mathbb{A}_2$. We can easily verify that the quasi–multiautomaton $\mathbb{A}_J$ is abelian (or commutation), cyclic and strongly connected.

Remark 3.5 *We can used the set $\mathbb{J}_C\mathbb{A}_2(J)$ instead of the set $\mathbb{J}\mathbb{A}_2(J)$ and corresponding results will be analogical. This can be easily verified.*

4 CONCLUSION

In connection with the above considerations there can be constructed multiautomata (with output), $\mathbb{A} = (A, S, B, \delta, \lambda)$, where A, B are (semi–)hypergroups of input, output symbols respectively. Further $\delta : A \times B \longrightarrow S$ is the transition (next–state) function satisfying GMAC and $\lambda : A \times S \longrightarrow B$ is the output function. Concrete interpretations of these structures will be objects of further investigations of the author.

These quasi–multiautomata are systems that can be used for the transmission of information of certain type. They are belonging to systems involving the modelling of various processes and the reciting of inner connections of time developable processes.

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