CONTROLLABILITY AND CONTROL CONSTRUCTION FOR A CERTAIN CLASS OF LINEAR MATRIX SYSTEMS WITH DELAY

Ganna Piddubna
Doctoral Degree Programme (2), FEEC BUT
E-mail: xpidd00@stud.feec.vutbr.cz

Supervised by: Jaromír Baštinec
E-mail: bastinec@feec.vutbr.cz

Abstract: In this paper existence of solutions of a certain class of differential linear matrix equations with delay was investigated. The solutions were found in general form. Necessary and sufficient condition for controllability of differential linear matrix equation with delay was defined and control was built. Paper contains calculated examples.

Keywords: matrix equation with delay, matrix exponential

1 INTRODUCTION

In many dynamic systems a delay appears. For example, in the simplest electronical circuits, the delay effect appears in voltage and current signals because of elements such as capacitors and inductors, respectively. Such dynamic systems can be described by systems of differential equations with aftereffects.
This paper is devoted to computing the solution of differential linear matrix equation with delay, described as follows, $\dot{X}(t) = AX(t) + AX(t-\tau)$. To solve this matrix equation, the "step by step method" has been used. The solution has been presented with help of the special matrix function - matrix exponential. Matrix exponential was used for solving differential equations by Krasovsky [7], [8] and for solving systems with aftereffects by many authors, e.g. Boichuk, Diblík, Khusainov, Růžičková, Shuklin [3] - [6].
The corresponding control problem has been built, a necessary and sufficient condition for controllability has been proposed and the control has been built.

2 LINEAR MATRIX EQUATION WITH DELAY

Let we have the equation

$$\dot{X}(t) = AX(t) + AX(t-\tau), \tag{1}$$

with initial condition

$$X(t) = I, \qquad -\tau \le t \le 0,$$

where A is square matrix, I is identity matrix, $\tau > 0, \tau \in R$ is a constant delay.

Definition 2.1 *Let A be a square matrix. Matrix exponential is defined by*

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!},$$

where I is the identity matrix.

Theorem 2.2 [2] *Let A is regular. Then the solution of equation* (1) *with identity initial condition has the recurrent form:*

$$X_{n+1}(t) = e^{A(t-n\tau)}X_n(n\tau) + \int_{n\tau}^{t} e^{A(t-s)}AX_n(s-\tau)ds,$$

where $X_n(t)$ *is defined on the interval* $(n-1)\tau \le t \le n\tau$.

Theorem 2.3 [1] *Let A is regular. Then the solution of equation* (1) *with identity initial condition has the form:*

$$X_k(t) = \sum_{l=o}^{k-1} 2e^{A(t-l\tau)} \sum_{p=0}^{l} (-1)^{p+l} A^p \frac{(t-l\tau)^p}{p!} + (-1)^k I,$$

where $X_k(t)$ *is defined on the interval* $(k-1)\tau \le t \le k\tau$.

Let we have the linear heterogeneous equation with delay

$$\dot{X}(t) = AX(t) + AX(t-\tau) + F(t). \tag{2}$$

If we have initial condition in the form

$$X(t) = \varphi(t), -\tau \le t \le 0, \tag{3}$$

where $\varphi(t) \in C^1[-\tau, 0]$, then we could write the following result.

Theorem 2.4 [2] *Let A is regular. The solution of heterogeneous equation* (2) *with the initial condition* (3) *has the form*

$$X(t) = \mathrm{X}_n(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_n(t-\tau-s)\varphi'(s)ds + \int_{0}^{t} \mathrm{X}_n(t-\tau-s)F(s)ds,$$

where $\mathrm{X}_n(t)$ *is the solution of the equation* (1) *with identity initial condition, defined in Theorem 2.3.*

3 CONTROLLABILITY OF THE LINEAR MATRIX SYSTEM WITH DELAY

3.1 GENERAL TERMS

Let X is the space of states of dynamic system; U is the set of the controlled effects (controls). Let $x = x(x_0, u, t)$ is the vector that characterizes state of the dynamic system in moment of time t, by the initial condition x_0, $x_0 \in X$, $(x_0 = x|_{t=t_0})$ and by the control function u, $u \in U$.

Definition 3.1 *The state* x_0 *is called controllable state in the class U (controlled state), if there are exist such control* $u(x_0) \in U$ *and the number* T, $t_0 \le T$ *that* $x(x_0, u(x_0), T) = 0$.

Definition 3.2 *If every state* $x_0 \in X$ *of the dynamic system is controllable, then we say that the system is controllable (controlled system).*

Consider the following Cauchy's problem:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Ax(t-\tau) + Bu(t),\ t \in [0, T],\ T < \infty, \\ &x(0) = x_0,\ x(t) = \varphi(t),\ -\tau \le t < 0, \end{aligned} \tag{4}$$

where $x = (x_1, ..., x_n)^T$ is the vector of phase coordinates, $x \in X$, $u(t) = (u_1(t), ..., u_r(t))^T$ is the control function, $u \in U$, U is the set of piecewise-continuous functions; A, B are constant matrices of dimensions $(n \times n)$, $(n \times r)$ respectively, τ is the constant delay.
Space of states Z of this system is the set of n-dimensional functions.

$$\{x(\theta),\, t-\tau \leq \theta \leq t\} \tag{5}$$

The space of the n-dimensional vectors x (phase space X) is subspace for Z. The initial state z_0 of the system (4) is determined by conditions

$$z_0 = \{x_0(\theta),\, x_0(\theta) = \varphi(\theta),\, -\tau \leq \theta < 0,\, x(0) = x_0\}. \tag{6}$$

The state $z = z(z_0, u, t)$ of the system (4) in the space Z in moment of time t is defined by trajectory segment (5) of phase space X.
Next considered, that the movement system (4) goes $(t \geq 0)$ in the space of continuous function. We determined initial state (6) of the function $\varphi(\theta)$ as piecewise-continuous.
In accordance with specified definitions, state (6) of the system (4) is controllable if there exist such control $u \in U$ that $x(t) \equiv 0$, $T - \tau \leq t \leq T$ when $T < \infty$.

3.2 THE CONSTRUCTION OF CONTROL FOR SYSTEM WITH DELAY

Let we have the control system of differential matrix equation

$$\dot{x}(t) = Ax(t) + Ax(t-\tau) + Bu(t), \quad x(t) \in R^n, \quad t \geq 0, \quad \tau > 0. \tag{7}$$

where $x(t) = \varphi(t), -\tau \leq t \leq 0, A, B$ are square constant matrices.

Remark 3.3 *For convenience purpose, here and further, we say that $x(t)$ is a vector of length n. All next statements are proved in the same way for the case when $x(t) = X(t)$ is a matrix of dimension $(n \times n)$.*

Theorem 3.4 [1] *For controllability of linear system with delay (7) is necessary and sufficient to next condition to hold: $t \geq (k-1)\tau$ and $rank(S) = n$, where*

$$S = \{B\ (AB)\ (A^2B)\ ...\ (A^{k-1}B)\ ...\},$$

hence S is a matrix which was achieved by recording matrices B, AB, ..., $A^{k-1}B$, ... side by side.

Theorem 3.5 [1] *Let $t_1 \geq (k-1)\tau$ and the necessary and sufficient condition for controllability is implemented:*

$$rank(S) = rank\Big(\{B\ (AB)\ (A^2B)\ ...\ (A^{k-1}B)\ ...\}\Big) = n.$$

Then the control function can be taken as

$$u(s) = [\mathrm{X}_0(t_1-\tau-s)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1-\tau-s)BB^T[\mathrm{X}_0(t_1-\tau-s)]^T ds\right]^{-1} \mu, \tag{8}$$

where

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds.$$

4 EXAMPLES

Let us consider few examples of controllability researches of the linear matrix systems with delay.

Example 4.1

Let us have the differential equation of 3-th degree with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t), \text{ where } A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and A is regular. We want to know if this system is controllable so let us check the necessary and sufficient condition. We will find the matrix S:

$$S = \{B\ (AB)\ (A^2B)\ \dots\ (A^{k+1}B)\ \dots\} = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 3 & 3 & 0 & & k & k & 0 & \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \dots & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & \end{pmatrix}$$

We have, $rank(S) = 2$, so the system is not controllable.

Example 4.2

Let us have the differential equation of 3-th degree with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t), \text{ where } A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and A is regular. It is easy to see that the necessary and sufficient condition for controllability is implemented (because of full rank of the matrix B, matrix S have full rank too), so the system is controllable.
Let us construct such control function, that move system in time moment $t_1 = 2$ in point $x_1 = (1,1,1)^T$, using initial condition $x_0(t) = \varphi(t) = (0,0,0)^T$, $-1 \le t \le 0$. Using the result of the theorem (3.5) we write:

$$u(t) = [\mathrm{X}_0(t_1 - \tau - t)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1 - \tau - s)BB^T [\mathrm{X}_0(t_1 - \tau - s)]^T ds \right]^{-1} \mu,$$

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s)ds.$$

While $\varphi(t) = (0,0,0)^T$, $-1 \le t \le 0$ then $\mu = (1,1,1)^T$. So, we have

$$u(t) = [\mathrm{X}_0(1-t)B]^T \left[\int_0^{2} \mathrm{X}_0(1-s)BB^T\, [\mathrm{X}_0(1-s)]^T\, ds \right]^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

While $t_1 = 2$, then $k = 2$ and, using (8) we can calculate

$$u(t) = \begin{pmatrix} 2(t+1)e^t + 2(t^2 - t - 1)e^{t-1} + 1 & 2e^t + 2(t-2)e^{t-1} & 0 \\ 2(t+1)e^t + 2(t^2 - t - 1)e^{t-1} + 1 & 2e^t + 2(t-2)e^{t-1} & 0 \\ (t^2 + 2t)e^t + (t^3 - 3t + 2)e^{t-1} & 2te^t + 2(t-1)^2 e^{t-1} & 2e^t + 2(t-2)e^{t-1} \end{pmatrix} \begin{pmatrix} 0.05 & -0.13 & 0.09 \\ -0.13 & 0.38 & -0.25 \\ 0.09 & -0.25 & 0.18 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$u(t) = 0.01 \begin{pmatrix} 2(t+1)e^t + 2(t^2 - t - 1)e^{t-1} + 1 \\ 2(t+1)e^t + 2(t^2 - t - 1)e^{t-1} + 1 \\ (t^2 + 2t + 4)e^t + (t^3 + t - 6)e^{t-1} \end{pmatrix}.$$

5 CONCLUSION

In this paper a solution of the system in general form was built. The necessary and sufficient condition for controllability of this system was defined and control was built. Two examples were given to illustrate the proposed theory. Getting results analogous to the ones in sections 2 and 2 for equation $\dot{X}(t) = AX(t) + BX(t-\tau)$, where A, B are different matrices, remains an open problem.

This research was supported by Grant FEKT-S-11-2-921 of Faculty of Electrical Engineering and Communication, BUT.

REFERENCE

[1] BAŠTINEC, J., PIDDUBNA, G.: *Controllability for a certain class of linear matrix systems with delay.* 11th International conference Aplimat. Bratislava, STU. 2012, 93 - 102. ISBN 978-80-89313-58-7.

[2] BAŠTINEC, J., PIDDUBNA, G.: *Solution of Matrix Linear Delayed System.* 7th conference of mathematics and physics at the technical universities with international participation. 2011. p. 48 - 57. ISBN 978-80-7231-815-5.

[3] BOICHUK, A., DIBLÍK, J., KHUSAINOV, D., RŮŽIČKOVÁ, M.,: *Boundary Value Problems for Delay Differential Systems*, Advances in Difference Equations, vol. 2010, Article ID 593834, 20 pages, 2010. doi:10.1155/2010/593834

[4] BOICHUK, A., DIBLÍK, J., KHUSAINOV, D., RŮŽIČKOVÁ, M.,: *Fredholm's boundary-value problems for differential systems with a single delay,* Nonlinear Analysis, **72** (2010), 2251–2258. (ISSN 0362-546X)

[5] BOICHUK, A., DIBLÍK, J., KHUSAINOV, D., RŮŽIČKOVÁ, M.,: *Controllability of linear discrete systems with constant coefficients and pure delay,* SIAM Journal on Control and Optimization, **47**, No 3 (2008), 1140–1149. DOI: 10.1137/070689085, url = http://link.aip.org/link/?SJC/47/1140/1. (ISSN Electronic: 1095-7138, Print: 0363-0129)

[6] DIBLÍK, J., KHUSAINOV, D., LUKÁČOVÁ, J., RŮŽIČKOVÁ, M., : *Control of oscillating systems with a single delay*, Advances in Difference Equations, Volume 2010 (2010), Article ID 108218, 15 pages, doi:10.1155/2010/108218.

[7] KRASOVSKII N.N.: *Inversion of theorems of second Lyapunov's method and stability problems in the first approximation*. Applied Mathematics and Mechanics. 1956. 255-265. (In Russian)

[8] KRASOVSKII N.N.: *The theory of motion control. Linear systems*. Nauka. 1968. 475. (In Russian)