APPROXIMATION OF TERRAIN DATA UTILIZING MULTIVARIATE SPLINES

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Abstract: The optimization of near-earth flight trajectories is built on the availability of high fidelity terrain data. Their fast, efficient and accurate computational evaluation is among the key implementation requirements. In order to meet the desired criteria a methodology of *multivariate simplex splines* is being introduced. This relatively new technique is based on a triangulation of the approximation domain by simplices. These are described by a Bernstein polynomial in barycentric coordinates. The continuity to the given order between neighboring simplices is achieved through definition of special conditions. The theory is general in the sense that it can be used in an n-dimensional Euclidean space. In this paper the method of simplex splines is demonstrated on a scattered bivariate dataset generated by the Mexican hat function.

Keywords: simplex spline, terrain data modelling, Bernstein polynomial, B-form, B-coefficients

1 INTRODUCTION

The problem of data modeling and especially scattered data modeling, is an important phenomenon within different areas of industrial application. The exact description of the different environments, as well as a fast evaluation of the immediate environmental changes is essential. The utilization of various mathematical methods for the assessment of a wide range of features of interest e.g. gradient based methods, assumes evaluation of a continuous differentiable function. This approach supports the need of mathematical modeling the observed data.

2 MULTIVARIATE SIMPLEX SPLINES

The first section introduces the basics behind the multivariate simplex spline theory. A more rigorous description may be found in the work of Lai and Schumaker [4] or of De Visser in his PhD. thesis [3].

2.1 PRELIMINARIES

Simplex spline function is a piecewise defined polynomial attached to structures called simplices. Simplex is a generalization of the notion of a triangle or tetrahedron to an arbitrary dimension. It can be formally defined as a convex hull:

$$t = \left\langle \mathcal{V}_t \right\rangle \in \mathbb{R}^n, \tag{1}$$

with t the n-simplex, $\mathcal{V}_t = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ the tuple of vertices and n the dimension. Every simplex also forms its own coordinate system known as barycentric coordinate system with point locations expressed relatively to a given simplex. Barycentric coordinates are important in simplex splines theory as the polynomials lined by simplices are defined in terms of barycentric coordinates. These polynomials are of a special type called Bernstein polynomials and often denoted in a B-form as follows:

$$p(\mathbf{b}) = \sum_{|k|=d} c_k B_k^d(\mathbf{b}), \tag{2}$$

with $p(\mathbf{b})$ an arbitrary polynomial, B_k^d Bernstein basis terms of degree d, $\mathbf{b}$ barycentric coordinate and most importantly c_k control coefficients or *B-coefficients*. The B-coefficients control the shape of a B-form polynomial by scaling individual basis functions. The key task in the modeling process is to find values of the B-coefficients for every simplex polynomial. It was proved by de Boor that any polynomial can be expressed in B-form [1].

2.2 TRIANGULATION AND CONTINUITY CONDITIONS

A subdivision of any geometric object into simplices is called triangulation. In a 2-dimensional case it lead to a subdivision into triangles, hence the name. In a formal definition, a triangulation $\mathcal{T}$ is a set consisting of J simplices defined as:

$$\mathcal{T} = \bigcup_{i=1}^{J} t_i,\ t_i \cap t_j \in \{\emptyset, \tilde{t}\},\ \forall (t_i, t_j) \in \mathcal{T},\ i \neq j \tag{3}$$

with $\tilde{t}$ a k-face of the n-simplices t_i and t_j, with $0 \leq k \leq n-1$ [3, section 2.3.2]. The fact that simplices are adjoined in triangulation makes simplex spline theory a very powerful concept. Modeling of scattered data is possible thanks to triangulation over randomly shaped domain. A value of simplex spline function for any given point located within simplex t depends only on the polynomial associated with this simplex. Therefore, the simplex spline function has a local basis which makes the evaluation very effecient.

The triangulation introduces a continuity problem among neighboring polynomials. The solution for this issue was suggested by de Boor [1]. Continuity conditions somehow link B-coefficients of neighboring polynomials and are formulated in a so-called smoothness matrix. This matrix is than used as a linear equality constraint in a linear regression scheme.

2.3 LINEAR REGRESSION

The final step during training process of a simplex spline function is an estimation of B-coefficients of all polynomials associated with simplices. One method proposed by de Visser in [3] using generalized linear regression is formulated as:

$$\mathbf{y} = \mathbf{X}\mathbf{c} + \mathbf{r} \text{ subject to } \mathbf{H}\mathbf{c} = 0, \tag{4}$$

with $\mathbf{y} \in \mathbb{R}^{N\times 1}$ all observations, $\mathbf{X} \in \mathbb{R}^{N\times(\hat{d}\cdot J)}$ the full-triangulation regression matrix for all observations, $\mathbf{c} \in \mathbb{R}^{(\hat{d}\cdot J)\times 1}$ the global B-coefficient vector, $\mathbf{r} \in \mathbb{R}^{N\times 1}$ a residual, with $\mathbf{H} \in \mathbb{R}^{(E\cdot R)\times(J\cdot\hat{d})}$ the smoothness matrix and with N number of observations, $\hat{d}$ number of B-coefficients per simplex, R the total number of continuity conditions per edge, and J, E the total number of simplices and edges in triangulation respectively. MATLAB provides many built-in functions to solve this kind of constrained linear least-squares optimisation problem e.g. `lsqlin`.

3 EXPERIMENTAL RESULTS

The methodology of the simplex splines as described above is demonstrated on a simple numerical experiment. The accuracy of the modeled data and the interpolation speed are mainly affected by three important function variables that need to be taken into account during the experimental setup. First is the number of elements of the input dataset, second is the degree of the polynomials defined on every simplex and third is the continuity order between adjoining polynomials. The last two form a structure called splines space on a given triangulation $\mathcal{T}$ denoted as $\mathcal{S}_d^r(\mathcal{T})$ and defined as:

$$\mathcal{S}_d^r(\mathcal{T}) := \{ s \in C^r(\mathcal{T}) \,:\, s|_t \in \mathbb{P}_d,\ \forall t \in \mathcal{T} \} \tag{5}$$

with $\mathbb{P}_d$ the space of all polynomials of degree d and with C^r the continuity order.

A set of 5000 data points $\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2\}$ was generated for the following example. The data generating function was defined by well known Mexican hat function. The Delaunay triangulation $\mathcal{T}$ was than applied on a regular grid overlying dataset $\mathcal{X}$ as can be seen in Figure 1 together with response function. This triangulation consists of $|\mathcal{T}| = 128$ triangles.

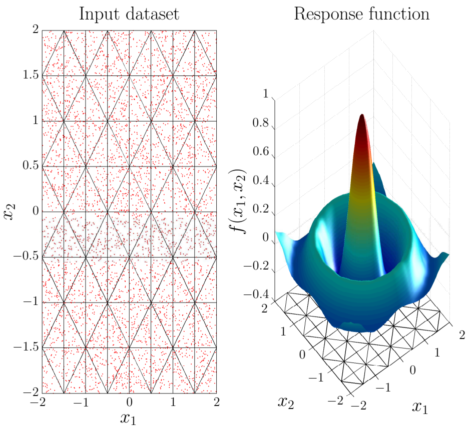

Figure 1: Delaunay triangulation $\mathcal{T}$ (left) and the Mexican hat function (right).

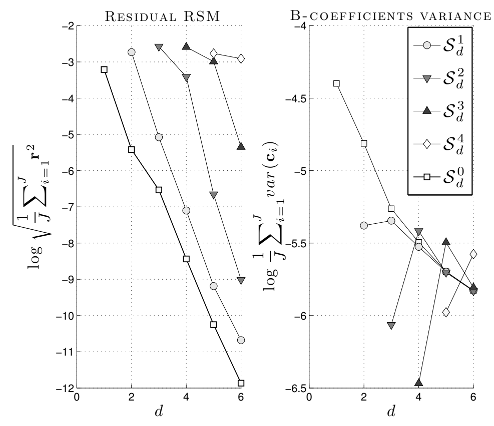

Figure 2: RMS of the residual $\mathbf{r}$ (left) and mean B-coefficients variance (right) for different spline spaces.

A number of simplex spline models in various spline spaces were trained during the presented experiment. As the linear regression was used to interpolate the identification dataset, it allowed to assess the quality of the model directly via the residual $\mathbf{r}$ from Eq(4). The left hand side in Figure 2 shows the RSM of the residual in a number of spline spaces. The increased degree of the polynomial leads to the expected reduction in the residual part. The right hand side of the same figure shows the mean variance of B-coefficients as proposed in [2]. It can be seen that the variance of B-coefficients is depending on the continuity order of the adjoining polynomial–with higher continuity order the B-coefficients of polynomials are more interdependent and their variance is decreasing. As a result, the approximation power of the simplex spline function is reduced by a higher continuity order. An experimentally suggested outcome states that in order to achieve a good approximation, the relation between polynomial degree d and continuity order r should maintain $d > 2r$.

4 CONCLUSION

Multivariate simplex splines were introduced into the framework of terrain data interpolation. The advantages of the presented technique feature scattered data modeling on non-rectangular domain, a computationally fast and efficient evaluation and mathematically suitable defined theory for n-dimensional spaces. The experimental implementation was made in MATLAB and demonstrated as shown, however, many other kinds of datasets can be approximated by the method of simplex splines e.g. more dimensional aerodynamic data, thermodynamic data, atmospheric data, etc.

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