STABILITY PROPERTY OF SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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ABSTRACT

In this paper, exponential-type stability of linear neutral delay-differential systems with constant coefficients is investigated using Lyapunov-Krasovskii type functionals more general that those reported in the literature. A delay-dependent sufficient conditions for the stability are formulated in terms of positivity of auxiliary matrices. Developed approach is used to characterize decay of solutions (by inequalities for the norm of arbitrary solution and its derivative) not only in the case of stability, but in a general case as well.

1 INTRODUCTION

In this paper we will give estimations of solutions of linear systems neutral differential equations with constant coefficients and a constant delay

$$\dot{x}(t) - D\dot{x}(t-\tau) = Ax(t) + Bx(t-\tau) \tag{1}$$

where $t \geq 0$ is an independent variable, $\tau > 0$ is a constant delay, A, B and D are $n \times n$ constant matrices and $x\colon [-\tau, \infty) \to \mathbb{R}^n$ is a column vector-solution. The derivative "$\cdot$" is understand as a left-hand derivative. Let $\varphi\colon [-\tau, 0] \to \mathbb{R}^n$ be a continuously differentiable vector-function. The solution $x = x(t)$ of the problem (**??**), (**??**) on $[-\tau, \infty)$ where

$$x(t) = \varphi(t),\ \dot{x}(t) = \dot{\varphi}(t),\ t \in [-\tau, 0] \tag{2}$$

we define in a classical sense (we refer, e.g. to [**?**]) as a continuous on $[-\tau, \infty)$ function continuously differentiable on $[-\tau, \infty)$ except for points τp, $p = 0, 1, \ldots$, and satisfying equation (**??**) everywhere on $[0, \infty)$ except for points τp, $p = 0, 1, \ldots$.
In the paper we find estimation of the norm the difference of arbitrary solution $x = x(t)$ of problem (**??**), (**??**) and the steady state $x(t) \equiv 0$ in arbitrary moment $t \geq 0$.
Let $\mathcal{F}$ be a rectangular matrix. We will use the matrix norm

$$\|\mathcal{F}\| := \sqrt{\lambda_{\max}(\mathcal{F}^T \mathcal{F})}$$

where a symbol $\lambda_{\max}(\mathcal{F}^T\mathcal{F})$ denotes the maximal eigenvalue of the corresponding symmetric positive semi-definite matrix $\mathcal{F}^T\mathcal{F}$. Similarly $\lambda_{\min}(\mathcal{F}^T\mathcal{F})$ denotes the minimal eigenvalue of $\mathcal{F}^T\mathcal{F}$. We will use the following vector norms

$$\|x(t)\| := \sqrt{\sum_{i=1}^{n} x_i^2(t)}, \quad \|x(t)\|_\tau := \sup_{-\tau\le s\le 0} \{\|x(s+t)\|\}, \quad \|x(t)\|_{\tau,\beta} := \sqrt{\int_{t-\tau}^{t} e^{-\beta(t-s)} x^2(s)\,\mathrm{d}s}$$

where β is a parameter.
In this paper we will use Lyapunov-Krasovskii functionals of a quadratic type depending on running coordinates as well as on their derivatives

$$V_0[x(t),t] = x^T(t)Hx(t) + \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)\,G_1x(s) + \dot{x}^T(s)G_2\dot{x}(s)\right]\mathrm{d}s \tag{3}$$

and

$$V[x(t),t] = e^{pt}V_0[x(t),t] \tag{4}$$

where x is a solution of (**??**), β and p are real parameters, $n \times x$ matrices H, G_1 and G_2 are positive definite, and $t > 0$. This permits to derive estimations of solutions of system (**??**) not only in the case of stability, but in the case of instability as well.
Results obtained are new and establish exponential decay of solutions in terms of norms $\|x(t)\|$ and $\|\dot{x}(t)\|$. Although many approaches in the literature are used to judge the asymptotical stability our approach, except others, not only determines whether system (**??**) is exponentially stable but also gives delay-dependent estimation solutions in terms of norms for both $\|x(t)\|$ and $\|\dot{x}(t)\|$.
To the best of our knowledge, applications of general functionals (**??**), (**??**) to study of stability and estimation of solutions of system (**??**) in outlined way have not been performed yet.

2 EXPONENTIAL STABILITY AND ESTIMATES OF CONVERGENCE OF SOLUTIONS

At first we give two relevant definitions of stability we work with:

Definition 1 The zero solution of the system of equations of neutral type (**??**) is called exponentially stable in the metric C^0, if there exist constants $N_i > 0$, $i = 1,2$ and $\mu > 0$ such that for arbitrary solution $x = x(t)$ of (**??**) the inequality

$$\|x(t)\| \le [N_1\|x(0)\|_\tau + N_2\|\dot{x}(0)\|_\tau]\exp(-\mu t)$$

holds for $t > 0$.

Definition 2 The zero solution of the system of equations of neutral type (**??**) is called exponentially stable in the metric C^1, if it is stable in the metric C^0 and moreover there exist constants $R_i > 0$, $i = 1,2$ and $\nu > 0$ such that for arbitrary solution $x = x(t)$ of (**??**) the inequality

$$\|\dot{x}(t)\| \le [R_1\|x(0)\|_\tau + R_2\|\dot{x}(0)\|_\tau]\exp(-\nu t)$$

holds for $t > 0$.

We will give estimation of solutions of linear system (**??**) on interval $(0,\infty)$ using functional (**??**). Then it is easy to see that an inequality

$$\lambda_{\min}(H)\|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2x(s)\right]\mathrm{d}s \le V_0\left[x(t),t\right]$$

$$\le \lambda_{\max}(H)\|x(t)\|^2 + \int_{t-\tau}^{t} e^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^T(s)G_2x(s)\right]\mathrm{d}s$$

holds on $(0,\infty)$. We will use an auxiliary $3n\times 3n$-dimensional matrix

$$S = S(\beta, H, G_1, G_2)$$
$$:= \begin{pmatrix} -A^TH - HA - G_1 - A^TG_2A & -HB - A^TG_2B & -HD - A^TG_2D \\ -B^TH - B^TG_2A & e^{-\beta\tau}G_1 - B^TG_2B & -B^TG_2D \\ -D^TH - D^TG_2A & -D^TG_2B & e^{-\beta\tau}G_2 - D^TG_2D \end{pmatrix}$$

depending on the parameter β and the matrices H, G_1, G_2. Except this we will use numbers

$$\varphi(H) := \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \varphi_1(H,G_1) := \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)}, \quad \varphi_2(H,G_2) := \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)}.$$

Now we give statements on stability of the zero solution of system (**??**) and estimations of convergence of solution. Proof of the first one is given in [**?**].

Theorem 1 *Let there exist a parameter $\beta > 0$ and positive definite matrices H, G_1, G_2 such that matrix S is also positive definite. Then the zero solution of system* (**??**) *is exponentially stable in the metric C^0. Moreover, for the solution $x = x(t)$ of problem* (**??**), (**??**) *inequality*

$$\|x(t)\| \le \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\tau\varphi_1(H,G_1)}\,\|x(0)\|_\tau + \sqrt{\tau\varphi_2(H,G_2)}\,\|\dot{x}(0)\|_\tau\right]e^{-\gamma t/2}, \tag{5}$$

holds where $\gamma = \min\left\{\beta, \dfrac{\lambda_{\min}(S)}{\lambda_{\max}(H)}\right\}$.

Theorem 2 *Let all assumptions of Theorem* **??** *hold. Let the matrix D be nonsingular and $\|D\|e^{\gamma\tau/2} < 1$. Then the zero solution of system* (**??**) *is exponentially stable in the metric C^1. Moreover, for the solution $x = x(t)$ of problem* (**??**), (**??**) *inequality*

$$\|\dot{x}(t)\| \le \left[\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(H,G_1)}\right)\right)\|x(0)\|_\tau\right.$$

$$\left. + \left(1 + M\sqrt{\tau\varphi_2(H,G_2)}\right)\|\dot{x}(0)\|_\tau\right]e^{-\varsigma t/2} \tag{6}$$

where $M = \|A\| + \|DA + B\|e^{\gamma\tau/2}(1 - \|D\|e^{\gamma\tau/2})^{-1}$ *and*

$$\varsigma = \min\left\{\gamma, \frac{2}{\tau}\ln\frac{1}{\|D\|}\right\}$$

hold on $[0,\infty)$.

3 EXAMPLE

We will investigate system (**??**) where $n = 2$, $\tau = 1$,

$$D = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 0.1 \\ 0.1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$

i.e., the system

$$\dot{x}_1(t) = 0.5\dot{x}_1(t-1) - x_1(t) + 0.1x_2(t) + 0.1x_1(t-1), \tag{7}$$

$$\dot{x}_2(t) = 0.5\dot{x}_2(t-1) + 0.1x_1(t) - x_2(t) + 0.1x_2(t-1), \tag{8}$$

with initial conditions (**??**). Set $\beta = 0.1$ and

$$G_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \qquad H = \begin{pmatrix} 2 & 0.1 \\ 0.1 & 5 \end{pmatrix}.$$

The following computations were performed by using MATLAB & SIMULINK R2009a software.

The eigenvalues of matrix G_1 are $\lambda_{\min}(G_1) = \lambda_{\max}(G_1) = 1$, the eigenvalues of matrix G_2 are $\lambda_{\min}(G_2) \doteq 0.5858$ and $\lambda_{\max}(G_2) \doteq 3.4142$, and the eigenvalues of matrix H are $\lambda_{\min}(H) \doteq 1.9967$ and $\lambda_{\max}(H) \doteq 5.0033$.

The matrix S takes the form

$$S \doteq \begin{pmatrix} 2.15 & -1.11 & -0.11 & 0.06 & -0.55 & 0.3 \\ -1.11 & 6.17 & 0.08 & -0.21 & 0.4 & -1.05 \\ -0.11 & 0.08 & 0.8948 & -0.01 & -0.05 & -0.05 \\ 0.06 & -0.21 & -0.01 & 0.8748 & -0.05 & -0.15 \\ -0.55 & 0.4 & -0.05 & -0.05 & 0.6548 & 0.6548 \\ 0.3 & -1.05 & -0.05 & -0.15 & 0.6548 & 1.9645 \end{pmatrix}.$$

Eigenvalues of S are $\lambda_1(S) \doteq 6.7377, \lambda_2(S) \doteq 2.2297, \lambda_3(S) \doteq 1.8651, \lambda_4(S) \doteq 0.8967, \lambda_5(S) \doteq 0.8352, \lambda_6(S) \doteq 0.1445$. Because all eigenvalues are positive, matrix S is positively definite.

All conditions of Theorem 1 are satisfied so the zero solution of system (**??**), (**??**) is exponentially stable in the metric C^0.

Further we have

$$\varphi(H) \doteq \frac{5.003}{1.9967} \doteq 2.5056, \quad \varphi_1(G_1, H) \doteq \frac{1}{1.9967} \doteq 0.5008,$$

$$\varphi_2(G_2, H) \doteq \frac{3.4142}{1.9967} \doteq 1.7099, \quad \gamma \doteq \min\left\{0.1, \frac{0.1445}{5.0033}\right\} \doteq 0.0289.$$

$$\|A\| \doteq 1.1, \quad \|B\| \doteq 0.1, \quad \|D\| \doteq 0.5, \quad \|DA + B\| \doteq 0.45, \quad M \doteq 2.0266, \quad \varsigma = \gamma.$$

Because we have $\|D\|e^{\gamma\tau/2} \doteq 0.5073 < 1$, then all conditions of Theorem 2 are satisfied so the zero solution of system (**??**), (**??**) is exponentially stable in the metric C^1.

Finally, from (**??**) and from (**??**) there follows, that the inequalities

$$\|x(t)\| \le \left[\sqrt{2.5056}\|x(0)\| + \sqrt{0.5008}\,\|x(0)\|_1 + \sqrt{1.7099}\,\|\dot{x}(0)\|_1\right] e^{-0.0289t/2},$$

$$\|\dot{x}(t)\| \leq \left[\left(0.2+2.0266\left(\sqrt{2.5056}+\sqrt{0.5008}\right)\right)\|x(0)\|_1+\left(1+2.0266\sqrt{1.7099}\right)\|\dot{x}(0)\|_1\right]$$
$$\times\, e^{-0.0289t/2}$$
$$\doteq [4.8421\,\|x(0)\|_1+3.65\,\|\dot{x}(0)\|_1]\, e^{-0.0289t/2}.$$

holds on $[0,\infty)$.
The matrix H can be modified. Not only above used variant of it is appropriate to state results on stability. It is easy to verify that for

$$H=\begin{pmatrix} 2 & 0.1 \\ 0.1 & k \end{pmatrix}$$

where $k \in \{3,\dots,10\}$ (preserving initial values for β, G_1 and G_2) Theorem 1 as well as Theorem 2 remain valid.
Similarly, mentioned results on stability can be proved in cases when

$$G_1=\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},\ k=\{2,\dots,5\}$$

(preserving initial values for β, H and G_2)
or

$$G_2=\begin{pmatrix} 1 & 1 \\ 1 & k \end{pmatrix},\ k=\{3,4,5\}$$

(preserving initial values for β, H and G_1).

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