REPRESENTATION OF SOLUTION OF LINEAR DISCRETE SYSTEM WITH CONSTANT COEFFICIENTS, A SINGLE DELAY AND WITH IMPULSES

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ABSTRACT

The purpose of this paper is to develop a method for the construction of solutions of linear discrete systems with constant coefficients, with pure delay and with impulses. Solutions are expressed with the aid of a special function called a discrete matrix delayed exponential.

1 INTRODUCTION

We use the following notation throughout this paper: For integers s, q, $s \leq q$ we define a set $\mathbb{Z}_s^q := \{s, s+1, \dots, q-1, q\}$. Similarly we define sets $\mathbb{Z}_{-\infty}^q := \{\dots, q-1, q\}$ and $\mathbb{Z}_s^\infty := \{s, s+1, \dots\}$. The function $[\,\cdot\,]$ is the greatest integer function.

Consider initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{1}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0 \tag{2}$$

where $m \geq 1$ is a fixed integer, $B = (b_{ij})$ is a constant $n \times n$ matrix, $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ and $\Delta x(k) = x(k+1) - x(k)$.

We add impulses $J_i \in \mathbb{R}^n$ to x at points having a form $i(m+1)+1$ where the index $i \geq 0$ is defined as $i = \left[\frac{k-1}{m+1}\right]$ for every $k \in \mathbb{Z}_0^\infty$, i.e., we set

$$x(i(m+1)+1) = x(i(m+1)+1-0) + J_i \tag{3}$$

and investigate the solution of the problem (1) – (3).

Before we deal with the solution of the problem (1) – (3), we will give the definitions and a theorem needed to solve our problem. We will also show an example to get a better understanding of the problem.

Definition 1.1. *For arbitrary integers n and k, we define the binomial coefficient:*

$$\binom{n}{k} := \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if} \quad n \geq k \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

In this paper, we use a special matrix function called a discrete function exponential. Such a discrete matrix function was first defined in [1], [2].

Definition 1.2. *For an $n \times n$ constant matrix B, $k \in \mathbb{Z}$ and fixed $m \in \mathbb{N}$, we define the discrete matrix delayed exponential e_m^{Bk} as follows:*

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta \quad \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I \quad \text{if} \quad k \in \mathbb{Z}_{-m}^{0}, \\ I + B \cdot \binom{k}{1} \quad \text{if} \quad k \in \mathbb{Z}_1^{m+1}, \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} \quad \text{if} \quad k \in \mathbb{Z}_{(m+1)+1}^{2(m+1)}, \\ \dots \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} + \dots + B^{\ell} \cdot \binom{k-(\ell-1)m}{\ell} \\ \qquad \text{if} \quad k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}, \ \ell = 0, 1, 2, \dots \end{cases} \tag{5}$$

where Θ is $n \times n$ null matrix and I is $n \times n$ unit matrix.

The Definition 1.2 of the discrete matrix delayed exponential can be shortened as

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I + \sum\limits_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j} & \text{if} \quad k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}, \ \ell = 0, 1, 2, \dots. \end{cases}$$

Next, Theorem 1.3 is proved in [1].

Theorem 1.3. *Let B be a constant $n \times n$ matrix. Then, for $k \in \mathbb{Z}_{-m}^{\infty}$,*

$$\Delta \mathrm{e}_m^{Bk} = B \mathrm{e}_m^{B(k-m)}. \tag{6}$$

2 REPRESENTATION OF SOLUTION OF INITIAL PROBLEM

The following example illustrates the influence of impulses on the solution and serves as a motivation for the formulation of a general case.

Example 2.1. *We consider a homogeneous particular case of* (1) *if $n = 1$, $B = b \neq 0$, $b \in \mathbb{R}$, $m = 3$ and $f(k) = 0$, $k \in \mathbb{Z}_0^{\infty}$ together with an initial problem* (2) *for $\varphi(k) = 1$, $k \in \mathbb{Z}_{-3}^{0}$ and with* *impulses* $J_i \in \mathbb{R}$ *at points* $i(m+1)+1 = 4i+1$ *where* $i \geq 0$, $i = \left[\frac{k-1}{m+1}\right] = \left[\frac{k-1}{4}\right]$:

$$\Delta x(k) = bx(k-3), \tag{7}$$

$$x(-3) = x(-2) = x(-1) = x(0) = 1, \tag{8}$$

$$x(4i+1) = x(4i+1-0) + J_i. \tag{9}$$

Rewriting the equation (7) *as*

$$x(k+1) = x(k) + bx(k-3)$$

and solving it by the method of steps, we conclude that the solution of the problem, can be written in the form:

$$x(k) = b^0\binom{k+3}{0} \qquad \text{if} \quad k \in \mathbb{Z}_{-3}^{0},$$

$$x(k) = b^0\binom{k+3}{0} + b^1\binom{k}{1} + J_0 b^0\binom{k-1}{0} \qquad \text{if} \quad k \in \mathbb{Z}_{1}^{4},$$

$$x(k) = b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + J_0\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1}\right] + J_1 b^0\binom{k-5}{0}$$
$$\text{if} \quad k \in \mathbb{Z}_{5}^{8},$$

$$\vdots$$

$$\begin{aligned}
x(k) = {} & b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + \cdots + b^{\ell}\binom{k-3(\ell-1)}{\ell} \\
& + J_0\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1} + b^2\binom{k-7}{2} + \cdots + b^{\ell-1}\binom{k-4-3(\ell-2)}{\ell-1}\right] \\
& + J_1\left[b^0\binom{k-5}{0} + b^1\binom{k-8}{1} + b^2\binom{k-11}{2} + \cdots + b^{\ell-2}\binom{k-8-3(\ell-3)}{\ell-2}\right] \\
& + J_2\left[b^0\binom{k-9}{0} + b^1\binom{k-12}{1} + b^2\binom{k-15}{2} + \cdots + b^{\ell-3}\binom{k-12-3(\ell-4)}{\ell-3}\right] \\
& + \cdots \\
& + J_i\left[b^0\binom{k-4(i+1)+3}{0} + b^1\binom{k-4(i+1)}{1} + b^2\binom{k-4(i+1)-3}{2}\right. \\
& \quad \left. + b^3\binom{k-4(i+1)-6}{3} + \cdots + b^{\ell-(i+1)}\binom{k-4(i+1)-3(\ell-(i+2))}{\ell-(i+1)}\right] \\
& \quad \text{if} \quad k \in \mathbb{Z}_{4(\ell-1)+1}^{4(\ell-1)+4},\ \ell = 0,1,2,\ldots,\ i = \left[\tfrac{k-1}{4}\right],\ i \geq 0.
\end{aligned}$$

The solution of the problem (7) – (9) *can be shortened to*

$$x(k) = \sum_{j=0}^{\ell} b^j\binom{k-3(j-1)}{j} + \sum_{q=0}^{i} J_q \sum_{j=0}^{\ell-(q+1)} b^j\binom{k-4(q+1)-3(j-1)}{j}, \tag{10}$$

for $k \in \mathbb{Z}_{4(\ell-1)+1}^{4(\ell-1)+4}$, $\ell = 0,1,2,\ldots,$ $i = \left[\frac{k-1}{4}\right]$, $i \geq 0$.

Theorem 2.2. *Let B be a constant $n \times n$ matrix, m be a fixed integer. Then the solution of the initial Cauchy problem with impulses*

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{11}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{12}$$

$$x(i(m+1)+1) = x(i(m+1)+1-0) + J_i, \quad J_i \in \mathbb{R}^n, \quad i \geq 0, \quad i = \left[\tfrac{k-1}{m}\right] \tag{13}$$

can be expressed in the form:

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) + \sum_{q=0}^{i} J_q \mathrm{e}_m^{B(k-(q+1)(m+1))} \tag{14}$$

where $k \in \mathbb{Z}_{-m}^\infty$.

Before we will prove the Theorem 2.2 we will introduce an auxiliary lemma.

Lemma 2.3. *Let a function $F(k,n)$ of two discrete variables be given. Then*

$$\Delta_k \left[\sum_{j=1}^{k} F(k,j)\right] = F(k+1,k+1) + \sum_{j=1}^{k} \Delta_k F(k,j). \tag{15}$$

Proof. (Theorem 2.2) We substitute (14) into the equation (11):

$$\begin{aligned}
\Delta x(k) &= \Delta\Bigg[\mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) \\
&\qquad + \sum_{q=0}^{i} J_q \mathrm{e}_m^{B(k-(q+1)(m+1))}\Bigg] \\
&= \Delta \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \Delta \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \Delta\sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) \\
&\qquad + \sum_{q=0}^{i} J_q \Delta \mathrm{e}_m^{B(k-(q+1)(m+1))} \\
&= [\text{according to the Theorem 1.3 and the Lemma 2.3}] \\
&= B\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} B\mathrm{e}_m^{B(k-m-m-j)}\Delta\varphi(j-1) + \mathrm{e}_m^{B((k+1)-m-(k+1))} f(k+1-1) \\
&\qquad + \sum_{j=1}^{k} B\mathrm{e}_m^{B(k-m-m-j)} f(j-1) + \sum_{q=0}^{i} J_q B\mathrm{e}_m^{B(k-m-(q+1)(m+1))} \\
&= B\Bigg[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-m-m-j)} f(j-1) \\
&\qquad + \sum_{j=k-m+1}^{k} \mathrm{e}_m^{B(k-m-m-j)} f(j-1) + \sum_{q=0}^{i} J_q \mathrm{e}_m^{B(k-m-(q+1)(m+1))}\Bigg] + \mathrm{e}_m^{B(-m)} f(k) \\
&= \left[\text{according to the Definition 1.2 is } \mathrm{e}_m^{B(-m)} = I \text{ and for } j \in \mathbb{Z}_{k-m+1}^k \text{ is } \mathrm{e}_m^{B(k-2m-j)} = \Theta\right]
\end{aligned}$$

$$
\begin{aligned}
&= B\left[e_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(k-m-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k-m} e_m^{B(k-m-m-j)} f(j-1) \right. \\
&\qquad \left. + \sum_{q=0}^{i} J_q e_m^{B(k-m-(q+1)(m+1))} \right] + f(k) \\
&= Bx(k-m) + f(k)
\end{aligned}
$$

Now we substitute (14) into the left-hand side $\mathcal{L}$ and right-hand side $\mathcal{R}$ of (13):

$$
\begin{aligned}
\mathcal{L} &= x(i(m+1)+1) \\
&= e_m^{B(i(m+1)+1)}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(i(m+1)+1-m-j)}\Delta\varphi(j-1) \\
&\qquad + \sum_{j=1}^{i(m+1)+1} e_m^{B(i(m+1)+1-m-j)} f(j-1) + \sum_{q=0}^{i} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))}, \\
\mathcal{R} &= x(i(m+1)+1-0) + J_i \\
&= e_m^{B(i(m+1)+1)}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(i(m+1)+1-m-j)}\Delta\varphi(j-1) \\
&\qquad + \sum_{j=1}^{i(m+1)+1} e_m^{B(i(m+1)+1-m-j)} f(j-1) + \sum_{q=0}^{i-1} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))} + J_i.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{q=0}^{i} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))} &= \sum_{q=0}^{i-1} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))} + J_i e_m^{B(i(m+1)+1-(i+1)(m+1))} \\
&= \sum_{q=0}^{i-1} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))} + J_i e_m^{B(-m)} \\
&= \sum_{q=0}^{i-1} J_q e_m^{B(i(m+1)+1-(q+1)(m+1))} + J_i
\end{aligned}
$$

it is obvious that $\mathcal{L} = \mathcal{R}$ and (13) holds.

□

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