GENERATED FUZZY IMPLICATIONS

Vladislav Biba
Doctoral Degree Programme (2), FEEC BUT
E-mail: xbibav00@stud.feec.vutbr.cz

Supervised by: Dana Hliněná
E-mail: hlinena@feec.vutbr.cz

ABSTRACT

An implication in MV-logic is a mapping $I : [0,1]^2 \to [0,1]$, which is an extension of the classical implication. There are many ways of constructing fuzzy implications. Best known are (S,N)-implications and R-implications. In this paper we deal with the other possibility, which use a function of a single variable. Such construction is well-known in case of the t-norms. We study properties of this type of implications and the connections between our implications and mentioned well-known classes.

1 INTRODUCTION

Fuzzy implications, which are a generalization of the classical two-valued implications to the multi-valued setting, play an important role in many applications. For instance, $(S,N-)$implications generalize the material implication from the classical logic with a $t-$conorm instead of disjunction, while $R-$implications obtained from a $t-$norm generalize the intuitionistic (residual) logic implication to the framework of fuzzy logic, whereas $QL-$implications are the fuzzy counterparts of quantum logic implication. Each of these families possesses many different properties.

Definition 1.1 *A unary operator $n : [0,1] \to [0,1]$ is called a fuzzy negation if, for any $x, y \in [0,1]$,*

- $x < y \Rightarrow n(y) \leq n(x)$,
- $n(0) = 1, n(1) = 0$.

The negation n is called a *strict negation* if and only if the mapping n is continuous and strictly decreasing. A strict negation is strong if it is an involution. Best known fuzzy negation is $N_s(x) = 1 - x$ called a *Standard negation*.

Best known extension of the conjunction is called a t*-norm*. (Note that t-norm is not the only possibility of extending classical conjunction.)

Definition 1.2 *A triangular norm (t-norm for short) is a binary operation on the unit interval* $[0,1]$*, i.e., a function* $T:[0,1]^2 \to [0,1]$ *such that for all* $x,y,z \in [0,1]$*, the following four axioms are satisfied:*
(T1) Commutativity $T(x,y) = T(y,x)$*,*
(T2) Associativity $T(x,T(y,z)) = T(T(x,y),z)$*,*
(T3) Monotonicity $T(x,y) \le T(x,z)$ *whenever* $y \le z$*,*
(T4) Boundary Condition $T(x,1) = x$*.*

Dual operator to a t-norm is called a *t-conorm*, denoted S. The operator S can be obtained as $S(x,y) = 1 - T(1-x,1-y)$. For more information about this topic, see [5].

In the literature, we can find several different definitions of fuzzy implications. We will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [4]. The readers can obtain some background by reading [2] and [6].

Definition 1.3 *A function* $I:[0,1]^2 \to [0,1]$ *is called a fuzzy implication if it satisfies the following conditions:*

(I1) *I is decreasing in its first variable,*

(I2) *I is increasing in its second variable,*

(I3) $I(1,0) = 0$*,* $I(0,0) = I(1,1) = 1$*.*

2 PROPERTIES AND CLASSES OF IMPLICATIONS

In following definition we describe some important properties of fuzzy implications.

Definition 2.1 *A fuzzy implication* $I:[0,1]^2 \to [0,1]$ *satisfies:*

(NP) *the left neutrality property, or is called left neutral, if*

$$I(1,y) = y; \quad y \in [0,1],$$

(EP) *the exchange principle if*

$$I(x,I(y,z)) = I(y,I(x,z)) \text{ for all } x,y,z \in [0,1],$$

(IP) *the identity principle if*

$$I(x,x) = 1; \quad x \in [0,1],$$

(OP) *the ordering property if*

$$x \le y \iff I(x,y) = 1; \quad x,y \in [0,1],$$

Definition 2.2 *Let* $I:[0,1]^2 \to [0,1]$ *be a fuzzy implication. The function* N_I *defined by* $N_I(x) = I(x,0)$ *for all* $x \in [0,1]$*, is called the natural negation of* I*.*

One of the well-known classes of implications is represented by (S,N)-implications, which are based on given t-conorm and negation N.

Definition 2.3 *A function $I : [0,1]^2 \to [0,1]$ is called an $(S,N)-$implication if there exist a t-conorm S and fuzzy negation N such that*

$$I(x,y) = S(N(x),y), \quad x,y \in [0,1].$$

If N is a strong negation, then I is called a strong implication.

The following characterization of $(S,N)-$implications is from [1].

Theorem 2.4 *(Baczyňski and Jayaram [1], Theorem 5.1) For a function $I : [0,1]^2 \to [0,1]$, the following statements are equivalent:*

- *I is an (S,N)-implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation N.*
- *I satisfies (I2), (EP) and N_I is a continuous (strict, strong) fuzzy negation.*

Another way of extending the classical binary implication operator to the unit interval $[0,1]$ uses the *residuation I* with respect to a left-continuous triangular norm T

$$I(x,y) = \max\{z \in [0,1]; T(x,z) \le y\}.$$

The following characterization of $R-$implications is from [4].

Theorem 2.5 *(Fodor and Roubens [4], Theorem 1.14) For a function $I : [0,1]^2 \to [0,1]$, the following statements are equivalent:*

- *I is an R-implication based on some left-continuous t-norm T.*
- *I satisfies (I2), (OP), (EP), and $I(x,.)$ is a right-continuous for any $x \in [0,1]$.*

3 GENERATED IMPLICATIONS

It is well-known that it is possible to generate t-norms from one variable functions. It means that it is enough to consider the one variable function instead of two variable function. Moreover, we can generate implications in a similar way as the t-norms. One of these possibilities is described in the next theorem. We use pseudo-inverse of the function f, which is defined for non-increasing function as $f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\}$, with the convention $\sup\emptyset = 0$.

Theorem 3.1 *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f : [0,1]^2 \to [0,1]$ which is given by*

$$I_f(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ f^{(-1)}(f(y^+) - f(x)) & \text{otherwise,} \end{cases}$$

where $f(y^+) = \lim\limits_{y \to y^+} f(y)$ and $f(1^+) = f(1)$ is a fuzzy implication.

Proof can be found in [3]. Now we turn our attention to the relations with (S,N)- and R-implications and our implications I_f.

Proposition 3.2 *Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing function such that $f(1)=0$, then I_f satisfies IP and NP. Moreover, f is continuous in $x=1$ if and only if I_f satisfies OP.*

Directly from Definition 2.1 and the following equivalence for the strictly decreasing function f:

$$f^{(-1)}(x_0)=1 \iff x_0 \le \lim_{x\to 1^-} f(x) = f(1^-),$$

we get the condition for IP and NP. If f is discontinuous in $x=1$, then $\exists x_0>0; f^{(-1)}(x_0)=1$. Now $\exists x,y\in[0,1]; x>y \wedge f(y^+)-f(x)<x_0$, which leads to $I_f(x,y)=1$ i.e. violation of OP. See [3] for closer explanation.

Proposition 3.3 *Let $f:[0,1]\to[0,\infty]$ be a continuous strictly decreasing function such that $f(1)=0$, then the implication I_f satisfies EP.*

Again, proof can be found in [3]. Since in previous the proposition we have the continuous function f and the EP is needed for both R- and (S,N)-implications, we assume continuous f also in the following proposition:

Proposition 3.4 *Let $f:[0,1]\to[0,c]$ be a continuous bounded function such that $f(1)=0$, then the negation N_{I_f} based on I_f is strong negation.*

The last theorem of this section is a corollary of previous propositions and Theorems 2.4, 2.5.

Theorem 3.5 *Let $f:[0,1]\to[0,\infty]$ be a continuous strictly decreasing function such that $f(1)=0$, then I_f is an R-implication given by some left-continuous t-norm, and more if $f(0)<\infty$, then I_f is an (S,N)-implication, too.*

Example 3.6 *Let us have a function $f:[0,1]\to[0,\infty]$, given by $f(x)=(1-x)^2$. The implication I_f is*

$$I_f(x,y)=\begin{cases}1 & x\le y,\\ 1-\sqrt{(1-y)^2-(1-x)^2} & \text{otherwise.}\end{cases}$$

This I_f is actually an R-implication given by the t-norm $T(x,y)=\max(0,1-\sqrt{(1-x)^2+(1-y)^2})$: An R-implication I is given by $I(x,y)=\max\{z\in[0,1]; T(x,z)\le y\}$. Since $T(x,1)=x$ if $x\le y$, obviously $I(x,y)=1$. In case $x>y$ we get

$$\max(0,1-\sqrt{(1-x)^2+(1-z)^2})\le y,$$

$$(1-x)^2+(1-z)^2\le(1-y)^2,$$

$$z=1-\sqrt{(1-y)^2-(1-x)^2},$$

therefore the R-implication I is our implication I_f. Note that T is an Archimedean t-norm with the generator f.

By the previous theorem, I_f is also an (S,N)-implication. Therefore there is a question, which t-conorm S and negation n produce this implication:
Because of the Theorem 2.4 and the axiom (S4) $S(x,0)=x$, we have that n is actually a natural negation given by I_f, i.e. $n(x)=1-\sqrt{2x-x^2}$. If the t-conorm S is given by

$$S(x,y)=1-\sqrt{\max((x-2)\cdot x+1-2y+y^2,0)},$$

mapping $S(n(x),y)$ is implication I_f. This t-conorm can be obtained using the natural negation of I_f and the t-norm T as

$$S(x,y)=n(T(n(x),n(y))).$$

4 CONCLUSIONS

Findings in the previous sections allows to make the following characterization of the implications I_f:

- Let f be a bounded and continuous function, then I_f is R- and (S,N)-implication.
- Let f be a continuous and not bounded, then I_f is R- but not (S,N)-implication.
- There exists (not-continuous) functions f violating EP. Therefore some I_f are neither R- nor (S,N)-implications.

The I_f implications are the best explored generated implications to-date, but there are also other possibilities to define the implication using the function of one variable. We continue our research on this topic.

REFERENCES

[1] BACZYŇSKI, M., BALASUBRAMANIAM, J.: *On the characterizations of (S,N)-implications* Fuzzy Sets and Systems 158 (2007), 1713–1727.

[2] BACZYŇSKI, M., BALASUBRAMANIAM, J.: *Fuzzy implications* (Studies in Fuzziness and Soft Computing, Vol. 231), Springer, Berlin (2008)

[3] BIBA, V., HLINĚNÁ, D.:*Generated fuzzy implications and known classes of implications* Acta Universitatis Matthiae Belii, to appear

[4] FODOR, J. C., ROUBENS, M.: *Fuzzy Preference Modeling and Multicriteria Decision Support* Kluwer Academic Publishers, Dordrecht 1994.

[5] KLEMENT, E. P., MESIAR, R., PAP, E.: *Triangular Norms* Kluwer, Dordrecht 2000.

[6] MAS, M., MONSERRAT, M., TORRENS, J., TRILLAS, E.: *A survey on fuzzy implication functions* IEEE Transactions on Fuzzy Systems 15 (6) 1107–1121, (2007).

[7] SMUTNÁ, D.:*On many valued conjunctions and implications* Journal of Electrical Engineering, 10/s, vol. 50, 1999, 8–10

[8] SCHWEIZER, B., SKLAR, A.: *Probabilistic Metric Spaces* North Holland, New York, 1983