# USING THE NEWTON TANGENT METHOD FOR LOCALIZATION POLYNOMIAL MULTIPLE ROOTS BY INCREMENT ARGUMENT METHOD


Ing. Petr SADOVSKÝ, Doctoral Degree Programme (4)
Dept. of Biomedical Engineering, FEEC, BUT
E-mail: petrsad@feec.vutbr.cz

Supervised by: Prof. Josef Diblík



## ABSTRACT

A long time period of the multiple roots searching using increment argument method combined with square overlap method led to new a research which would result in faster algorithm for multiple roots searching which used increment argument method.


## 1 INTRODUCTION

As the theoretical base of algorithm presented serves the so-called argument principle. This principle is known from theory complex functions. Let us describe it shortly (see e.g. [1]).

Let us suppose, that a complex function $w = F(z)$ is analytic in a simple connected domain $D$ bounded by contour $C$ and have no zero points on it. Then the number of roots $N$ of the equation

$$F(z) = 0, \tag{1}$$

inside $D$ equals to the absolute value of increment of argument on the curve, which is created by the function $w = F(z)$ as independent variable $z$ traverses $C$ divided by magnitude $2\pi$:

$$N = \frac{1}{2\pi}|\Delta_C \mathrm{Arg} f(z)|.$$

In other words: number of roots of equation (1) in $D$ equals the number of revolutions about the origin of coordinates of the vector $w = F(z)$ which runs from point $w = 0$ to point $w = F(z)$ completes as point $z$ traverses $C$ (the number of revolutions is assumed to be positive if $w$ revolves in the positive sense and negative if otherwise).

## 2 ALGORITHM

The algorithm for searching of multiple roots is possible to split into two parts. The first part finds out, if there are roots in one or rectangular region of complex plane (and determine their number). The second part (utilizing the first one) searches for algorithm, which is able to find out all the roots of polynomial.

### 2.1 FUNCTION FOR DETECTION OF A ROOT

For searching of roots of a polynomial was designed a function which gives back the number of roots in a given rectangle. This function uses the above described increment argument principle and calculate the number of revolutions around the coordinate origin of the complex plane when point $z$ traverses a rectangle.

### 2.2 LOCATION OF A ROOTS

The roots of a polynomial are always located in an definite part of the complex plane. The searching for them in the whole complex plane is not only impossible but also totally needless. It is possible to estimate positions of roots with the aid of following theorem.
**Theorem (location of roots)** For roots $x_1$, $x_2$, $\ldots$, $x_n$ of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0, \tag{2}$$

with $a_n \neq 0$, the following inequalities hold:

$$\frac{|a_0|}{|a_0| + B} \leq |x_k| \leq \frac{|a_n| + A}{|a_n|}, \tag{3}$$

$k = 1, \ldots, n$, where $A = \max(|a_{n-1}|, \ldots, |a_0|)$ and $B = \max(|a_n|, \ldots, |a_1|)$.

The inequalities (3) define the area between two circles. The searching algorithm simply discounts the inner area inside the inner circle because this circle only denotes the place, where are no roots. Roots searching area is realized from boundary circle which circumscribed a square, whose center is in the coordinate origin and its radius is

$$r = \frac{|a_n| + A}{|a_n|}. \tag{4}$$

Square region containing roots, is given by four points:

$$z_1 = -r - r \cdot i,\ z_2 = r - r \cdot i,\ z_3 = r + r \cdot i,\ z_4 = -r + r \cdot i. \tag{5}$$

### 2.3 FUNCTION FOR DESTINATION OF A ROOTS

For the location of the position of the roots by the method of increment argument principle were tested several methods. First as the most interesting of these methods was chosen square overlap method. But this method worked with huge redundancy of computation, because it was possible that in some cases the argument increment algorithm lost a root. (For details see [2].)

Second was tried out the Newton Tangent Method. This method was able to localize roots very quickly.

The known Newton recursive iteration formule (6) generates sequence which converge to one of the roots in complex plane.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{6}$$

The root which will be localized by this sequence depends on a basepoint in the complex plain.

For example the polynomial of fifth order

$$x^5 + 2x^4 + 2x^3 = 0.$$

This polynomial have five roots, one is complex-associate root and the other is three times multiple root.

$$\begin{array}{rcrcl} x_1 & = & -1 & + & 1 \cdot i \\ x_2 & = & -1 & - & 1 \cdot i \\ x_{3,4,5} & = & 0 & & \end{array}$$

There is a part of complex plain which is determinated by (4) and (5) in Figure 1. There are basepoints for the Newton recursive formula (6) at every side of the square (along area border). In Figure 1 are shown iterations (sequence which converting to roots given by Newton recursive iteration formula (6)) decreasing to some root.

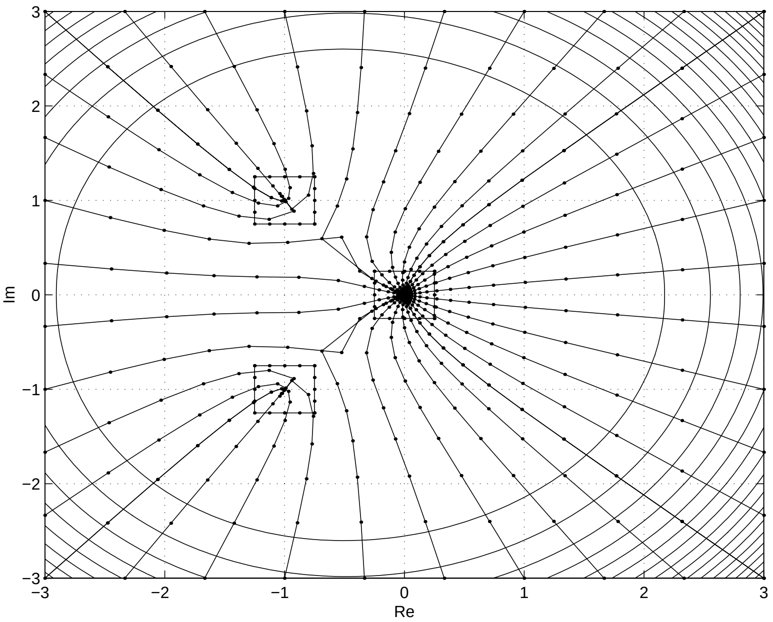


Figure 1: Localization of the roots of the polynomial $x^5 + 2x^4 + 2x^3 = 0$ in the complex plain.

After the first estimation of the roots in the complex plain are constructed squares around the located roots and increment argument algorithm is used for multiplicity searching. To find the roots more accurately the Newton tangent algorithm is used again. This situation is illustrated in Figure 2.

It is possible that some iterations leave the area inclosed by a square border and aim to the other root. These iterations are forgotten.

When root is being found with sufficient accuracy, geometric mean of the end points of iterations returns the root more precisely.

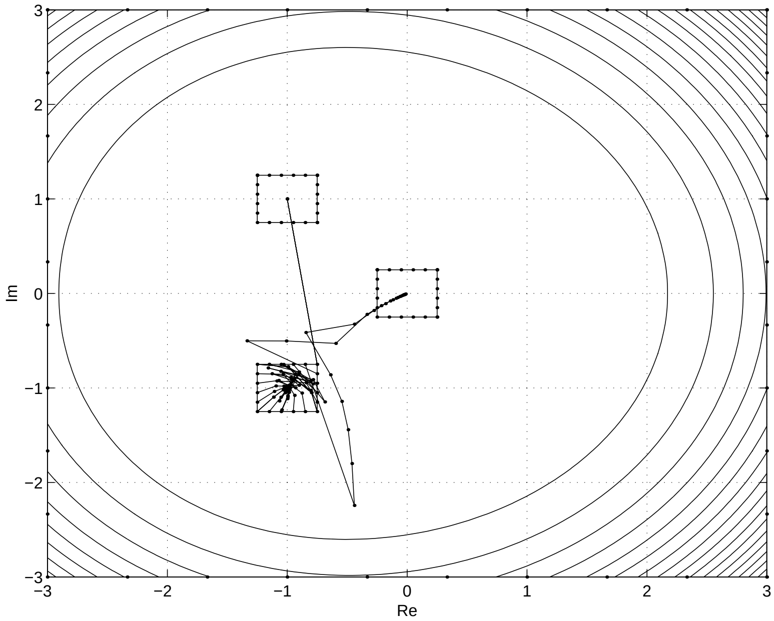


Figure 2: Localization of the roots of the polynomial $x^5 + 2x^4 + 2x^3 = 0$ in the complex plain.

## 3 RESULTS

The algorithm had been tested on several polynomials with multiple roots. It is confirmed, that this method is able to find multiple roots of a polynomial or roots, which are very close each to other. Nevertheless a sufficient accuracy in the searching of the roots was not reached still. The accuracy of roots location decreases when the multiplicity of the tested polynomial root is high. This phenomenon is explained below.

For the purpose of a demonstration of algorithm, let us consider the following polynomial equation

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0. \tag{7}$$

This polynomial have multiple root

$$x_{1,2,3,4,5} = -1.$$

The reason why this new method for searching of the multiple roots has been developed, is evident from following results.

To compare the results of used methods searching for the roots of the polynomial (7) were calculated roots by a function implemented by default in the MATLAB (8). This function is using the matrix decomposition. The MATLAB function found the root like a group of the close roots.

$$\begin{array}{rrcl} x_1 = & -1,0008 & & \\ x_2 = & -1,0003 & + & 0,0008\cdot i \\ x_3 = & -1,0003 & - & 0,0008\cdot i \\ x_4 = & -0,9993 & + & 0,0005\cdot i \\ x_5 = & -0,9993 & - & 0,0005\cdot i \end{array} \tag{8}$$

The algorithm which used increment argument method found 5 times root.

$$z_{1,2,3,4,5} = -0,9999 + 0,0000i. \tag{9}$$

It is obvious that multiple root was found by an algorithm which used increment argument method. Method implemented in MATLAB found multiple root like a group of the close roots.

## CONCLUDING REMARKS

Our results indicate that the using of the increment argument method combined with the Newton tangent method is possible and convenient. It is obvious from our results, that this method can find the roots with approximately equal results like the other methods, but our algorithm reached better results in multiple roots. The conclusion is, that the increment argument method is able to find multiple roots much more accurately then the other methods.

This method have also some disadvantages. Increment argument method have too complicated algorithm which causes that this method is rather slower then the others. The more multiple root we have, the more slower the computing is.

The roots which are too close to each other in the complex plain limit the possibilities of the algorithm of increment argument method. In this case, the loss of a root is also possible.

We assume, that for further use of this algorithm, will be useful to pre-set some software able to offer several solutions with graphic display of the situations.

## ACKNOWLEDGMENT

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