VECTOR FIELDS FOR PIECEWISE-LINEAR SYSTEM

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ABSTRACT

Description of real physical systems is often based on mathematical models with smooth and continuous functions. It come to reason that it is impossible to find general analytic solution. The piecewise linear approximation seems to be useful tool for solving this problem. The methods of building a proper mathematical models and explicit solutions are presented in this paper.

1 INTRODUCTION

For the majority physical systems it is not possible to express local analytic solution of corresponding equations. Our approach is based on piecewise linear (PWL) approximation. The space of results is decomposed into the regions separated by means of n-1 dimensional hyperplanes. Inside the segment we get the affine system, in which we can even formulate symbolic solution. This procedure was successfully used in the chaos theory [1]. Let suppose the following PWL dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

with continuous vector field **F**. We are interested in it´s existence and uniqueness with respect to the solution of state equation (1). Based on Pikard-Lindelot´s theorem it is possible to show that if the function is continuous on the open compact set then it fulfil the Lipschitz condition and there exist the neighborhood such as initial problem (1) has unique solution.

Obviously, for the system like this we can generate formal model shown on Fig. 1a. Single value condition deals with necessary continuous VA characteristics of nonlinear part.

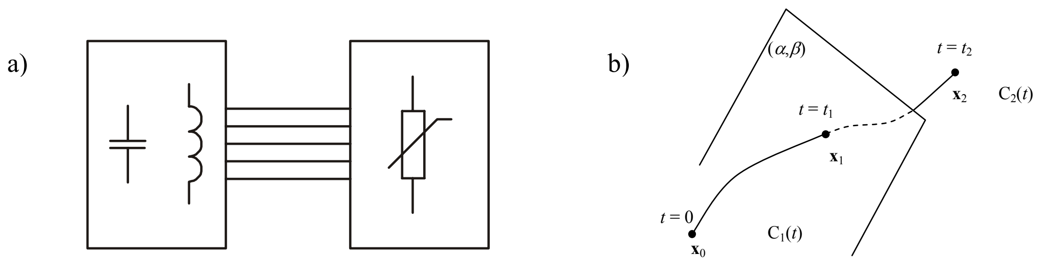

Fig. 1: *a) Formal model, b) Trajectory in the state space.*

2 MATHEMATICAL MODEL OF PWL FUNCTION

PWL approximation consists in substitution of nonlinear function by finite number of linear segments. However, very few practically useful methods are well known from the literature handling with approximation tasks. PWL function can be also obtained directly by using Čebyšev approximation. There is no algorithmic method for higher-dimensional functions [3]. Partial solution for weakly global definition domain of explicit function $\mathbf{f}: R^n \to R^1$

$$\mathbf{f}(\mathbf{x}) = a + B\mathbf{x} + \sum_{i=1}^{\sigma} c_i \left| \langle \alpha_i, \mathbf{x} \rangle - \beta_i \right|, \tag{2}$$

is given in [3].

For modeling electrical systems it is advantageous to use other form of description based on state variables u and j which matches condition

$$\mathbf{u} \cdot \mathbf{j} = \mathbf{0}, \mathbf{u} \geq \mathbf{0}, \mathbf{j} \geq \mathbf{0}. \tag{3}$$

Assume that gate voltages and currents of nonlinear blocks are composed by vectors **x**, **y** $\in R^n$. It is possible to re-formulate model in implicit form [2] as

$$\mathbf{y} = \mathbf{Au} + \mathbf{Bj} + \mathbf{f}, \tag{4}$$

$$\mathbf{x} = \mathbf{Cu} + \mathbf{Dj} + \mathbf{g}, \tag{5}$$

$$\mathbf{0} = \mathbf{Mu} + \mathbf{Nj} + \mathbf{q}, \tag{6}$$

where $\mathbf{A} \in R^{m \times k}$, $\mathbf{B} \in R^{m \times k}$, $\mathbf{f} \in R^m$, $\mathbf{C} \in R^{n \times k}$, $\mathbf{D} \in R^{n \times k}$, $\mathbf{g} \in R^n$, $\mathbf{M} \in R^{(k-n) \times k}$, $\mathbf{N} \in R^{(k-n) \times k}$ and $\mathbf{q} \in R^{(k-n)}$. For the given **x** it is necessary to compute vector **y**. This could be done by solving the linear complementarity [4] problem. For certain circumstances we can write following matrix equations

$$\mathbf{y} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{f}, \tag{7}$$

$$\mathbf{j} = \mathbf{Cx} + \mathbf{Du} + \mathbf{g}, \tag{8}$$

where $\mathbf{A} \in \mathrm{R}^{m \times n}$, $\mathbf{B} \in \mathrm{R}^{m \times k}$, $\mathbf{f} \in \mathrm{R}^m$, $\mathbf{C} \in \mathrm{R}^{k \times n}$, $\mathbf{D} \in \mathrm{R}^{k \times k}$ and $\mathbf{g} \in \mathrm{R}^k$. First step deals with deriving of the state vector **j** from (5), (6) and rewrite it into the form (8). Then we can express corresponding matrices as

$$\mathbf{C} = \begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_\mathrm{d} \\ \mathbf{0} \end{bmatrix}, \mathbf{D} = -\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C'} \\ \mathbf{M} \end{bmatrix}, \mathbf{g} = -\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{g'} \\ \mathbf{q} \end{bmatrix}, \tag{9}$$

where $\mathbf{I}_\mathrm{d}$ is n^{th} order unity matrix. Putting **j** into the inner and outer block (4), (5) of the decomposed model and recasting it in accordance with (7) we get remaining matrices

$$\mathbf{A} = \mathbf{B'}\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_\mathrm{d} \\ \mathbf{0} \end{bmatrix}, \mathbf{B} = \mathbf{A'} - \mathbf{B'}\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C'} \\ \mathbf{M} \end{bmatrix}, \mathrm{f} = \mathbf{f'} - \mathbf{B'}\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{g'} \\ \mathbf{q} \end{bmatrix}. \tag{10}$$

It is obvious that matrix $\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}$ must be regular. If this restriction is not holded, we can make equivalent changes for the purpose to reach the regularity. Efficiency of this operation is qualified by necessary n-dimensional definition domain, that means $dim(D) = n$, $D \subseteq R^n$. Let suppose that we want achieve existence of state $\mathbf{s}(2^k\text{-}1)$ ($\mathbf{j}_i > 0$, $i = 1, 2, \ldots$, k for new state variables) by means of changing the state pairs. Such transformation always exists because $dim(D) = n$, that´s why at least one PWL relation segment must exists. Assume the matrix $\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix} \in R^{k\times k}$ in the form

$$\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix} = \left[\begin{array}{c|ccc} \mathbf{D'}_n & & \mathbf{0} & \\ \hline & 1 & & \\ \mathbf{0} & & \ddots & \\ & & & 1 \end{array}\right] \quad (11)$$

where $\mathbf{D'}_n \in R^{n\times n}$. From previous analysis follows, that type of PWL relation (existence of partial state pairs combination respectively) is determined by inner block of the model, namely by matrices **M**, **N**, **q**. Inner block for the state $\mathbf{s}(2^k\text{-}1)$ represents $k - n$ equations for k variables **j**, which describes n-dimensional subspace. These equations are independent and then there is a chance to find in N an unity submatrix of dimension $k - n$, (11). If the state variables $\mathbf{j}_1,\ldots,\mathbf{j}_n$ equals 0, then the term $\mathbf{j}_{n+i} = \mathbf{-q}_i > 0$ for $i = 1,2,\ldots,k\text{-}n$ is valid. It is obvious that for small enough $\mathbf{j}_1,\ldots, \mathbf{j}_n$ the complementary condition shall be fulfiled, because solution of linear system depends continuously on the right side. Components $\mathbf{j}_1,\ldots, \mathbf{j}_n$ are embossed as parameter. For the regular matrix $\begin{bmatrix} \mathbf{D'} \\ \mathbf{N} \end{bmatrix}$ situation (11) it is sufficient submatrix $\mathbf{D'}_n$ to be regular. Then for $\mathbf{s}(2^k\text{-}1)$ we get

$$\mathbf{x} = \mathbf{D'}_n[\mathbf{j}_1,\ldots,\mathbf{j}_n]^{\mathrm{T}} + \mathbf{const}. \quad (12)$$

Matrix $\mathbf{D'}_n$ is regular and then $span\{\mathbf{D'}_n\} \subseteq R^n$. The definition domain can not has lower dimension since one segment of the definition domain in variables **x** for the combination $\mathbf{s}(2^k\text{-}1)$ has a dimension n. For a given **x** it is required to resolve the equation (8)

$$\mathbf{j} = \mathbf{D}\cdot\mathbf{u} + \mathbf{q}. \quad (13)$$

This term already contains the linear complementarity problem. In the case of matrix **D** satisfy the restriction $\forall x \neq 0 \;\; \exists k$ such as $x_k \cdot (Dx)_k > 0$, then it has (for a given **x**) unique solution. This solution can be expressed explicitly [4]. Assume $\mathbf{H}(\mathbf{x})$ defined as $\mathbf{H}(\mathbf{x}) = 1/2.(|\mathbf{x}| + \mathbf{x})$. In the case $\mathbf{D} \in R^{1\mathrm{x}1}$ the proper solution is

$$\mathbf{j} = \mathbf{H}(\mathbf{q}),\ \mathbf{u} = \mathbf{H}(-\mathbf{q}/\mathbf{D}), \quad (14)$$

For the rest of the systems (higher dimensional) we can find results using hierarchical procedure.

$$\begin{bmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_\alpha & \mathbf{D}_\beta \\ \mathbf{D}_\gamma & \mathbf{D}_\delta \end{bmatrix} . \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}. \quad (15)$$

Let´s define

$$\tilde{\mathbf{j}} = \mathbf{D}_\delta \cdot \tilde{\mathbf{u}} + \mathbf{q}_1 , \tag{16}$$

where $\tilde{u}$ and $\tilde{j}$ are corresponding parts of the vectors **u** and **j**, then

$$\tilde{\mathbf{j}}_1 = \mathbf{H}(\mathbf{D}_\delta \cdot \tilde{\mathbf{u}} + \mathbf{q}_1). \tag{17}$$

Explicit solution is preferable, mostly because of it doesn´t require matrix iteration operations. We can employ Katzenelson´s or Lemke´s algorithms.

3 PHASE SPACE TRAJECTORIES

The state models leads to dynamical complementary systems of the form [3]

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{f},\ \mathbf{j} = \mathbf{Cx} + \mathbf{Du} + \mathbf{g},\ \mathbf{u} \geq \mathbf{0}, \mathbf{j} \geq \mathbf{0}, \mathbf{u}^{\mathrm{T}} \mathbf{j} = 0, \tag{18}$$

which generally does not satisfy Lipschitz condition. In this case, the existence and uniqueness is not guaranteed. Let assume the problem of local solution of continuous PWL dynamical system. It does not matter which model is used, dynamical system is described by affine equation on the certain i th segment of the state space

$$\dot{\mathbf{x}} = \mathbf{A}_i \mathbf{x} + \mathbf{b}_i , \tag{19}$$

where $\mathbf{A}_i \in R^{n \times n}$, $\mathbf{x}, \mathbf{b}_i \in R^n$ with equilibrium point $\mathbf{x}_{Qi} = -\mathbf{A}_i^{-1} \mathbf{b}_i$. Equilibria outside the corresponding phase space segment are called to be virtual. General solution of the linear equation can be expressed as a linear combination of n independent solutions $x_1(t)$, $x_2(t)$, ... , $x_n(t)$ [4].

$$\mathbf{x}(t) = \sum c_j \mathbf{x}_j(t) = \mathbf{C}(t)\mathbf{c} , \tag{20}$$

where $C(t)$ is the so called fundamental matrix and $\mathbf{c} \in R^n$ is vector of initial conditions $x(0) = x_0$, concretely

$$\mathbf{c} = \mathbf{C}(0)^{-1} \mathbf{x}_0 . \tag{21}$$

For mentioned afinne system we get

$$\mathbf{x}_i(t) = \mathbf{C}_i(t)\mathbf{C}_i(0)^{-1}(\mathbf{x}_0 - \mathbf{x}_{Qi}) + \mathbf{x}_{Qi} . \tag{22}$$

For the case of simple (non-multiple) eigenvalues λ it is possible to rewrite matrix **C** as

$$\mathbf{C}_{\bullet,j} = e^{\lambda_j t} \mathbf{v}_j , \tag{23}$$

where $\mathbf{C}_{\bullet,j}$ denotes j th column of the matrix **C** and $\mathbf{v}_j$ is appropriate eigenvector. Fundamental matrix can be recasted into to form $\mathbf{C}_i(t)\mathbf{C}_i(0)^{-1} = e^{\mathbf{A}t}$. In the case of complex conjugated eigenvalues we get eigenvectors with complex entries. For two complex conjugated eigenvalues we are allowed to write the oscillating result as

$$\begin{aligned} \mathbf{x}_i(t) &= e^{\mathrm{Re}(\lambda_i)t}\left[\mathrm{Re}(\mathbf{v}_i)\cos(\mathrm{Im}(\lambda_i)t) - \mathrm{Im}(\mathbf{v}_i)\sin(\mathrm{Im}(\lambda_i)t)\right], \\ \mathbf{x}_{i+1}(t) &= e^{\mathrm{Re}(\lambda_i)t}\left[\mathrm{Re}(\mathbf{v}_i)\sin(\mathrm{Im}(\lambda_i)t) + \mathrm{Im}(\mathbf{v}_i)\cos(\mathrm{Im}(\lambda_i)t)\right]. \end{aligned} \tag{24}$$

The stitching method can be employed to obtain complete state space trajectory, by merging individual linear segments together. This is shown on Fig. 2. Assume transient area between region with index 1 and with index 2. Moreover, assume that there exists such a time t_1 when the trajectory intersects border plane in the point

$$\mathbf{x}_1 = \mathbf{C}_1(t_1)\mathbf{C}_1^{-1}(0)(\mathbf{x}_0 - \mathbf{x}_{Q1}) + \mathbf{x}_{Q1} . \tag{25}$$

The point x1 represents the initial conditions of the trajectory in the second segment

$$\mathbf{x}_2(t) = \mathbf{C}_2(t - t_1)\mathbf{C}_2^{-1}(0)(\mathbf{x}_1 - \mathbf{x}_{Q2}) + \mathbf{x}_{Q2} = \mathbf{C}_2(t)\mathbf{C}_2^{-1}(t_1)(\mathbf{x}_1 - \mathbf{x}_{Q2}) + \mathbf{x}_{Q2} . \tag{26}$$

If the boundary is given as   then for intersection we derive immediately

$$\alpha^T \mathbf{C}_1(t_1)\mathbf{C}_1^{-1}(0)(\mathbf{x}_0 - \mathbf{x}_{Q1}) + \mathbf{x}_{Q1} = \beta . \tag{27}$$

4 CONCLUSION

This paper deals with modeling of PWL phase state problems. It suggest the possibilities of transformation from implicit to LCP model characterization.

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