4D VOLUME CONSERVING AUTONOMOUS CHAOTIC OSCILLATOR

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ABSTRACT

Recently, the possibility to associate a dynamical system with solvable and semisimple groups in terms of Lie algebra theory has been demonstrated. In this paper, the new chaotic system based on Nosé-Hoover dynamics is derived by using a matrix Levi decomposition. The circuitry realization is experimentally tested via PSpice simulator.

1 INTRODUCTION

Since the chaos was confirmed in many disciplines of common life, it becomes a topic of increasing interest immedietely. The most unexpected chaotic solutions were observed in engineering applications for the first time. The behavior of dynamical system is given by it´s state equations and initial conditions. The Poincaré-Bendixon´s theorem stands that the state space of chaotical systems needs at least three dimensions. Many research on the theory of differential equations and 3rd order dynamical systems uncovers several distinct cases of algebraically simple chaotic flows. There are some reasons to believe that chaos can not be exhibited by a dynamical system with less than five terms including two nonlinearities or less than six terms with single nonlinearity. A systematic examination shows that both groups of autonomous dynamical systems can be recasted into the compact matrix form $\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{x} \cdot \mathbf{x}^{\mathrm{T}} \cdot \mathbf{C} + \mathbf{D}$, where **A** and **B** are 3×3 matrices, **C** and **D** are 1×3 vectors. Algebraical simplicity implies that **B** is thin with maximally two nonzero elements. These systems mostly offers two equilibrium points, but huge numerical searching [1] reveals the cases with single equilibrium point and surprisingly even without any equilibria. Elemental mathematical description and expansion of this system into higher dimensions is demonstrated in the next chapter. Second chapter shows the circuitry implementation of a given system with respect to the transparency of the realized set of ordinary differential equations. Recently, the novel networks realizing hyperchaotic flows are studied extensively. Equivalent flows can be generated by two two-dimensional subsystems coupled via a very simple piecewise linear scalar functions [4]. Unique property of hyperchaos is in strong resistance to dynamics reconstruction, factually to synchronization between transmitter and receiver side of communication channel.

2 MATHEMATICAL PRELIMINARY

Nosé-Hoover dynamical system has the matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1 }$$

Although this system produces the limit cycles, equation $\dot{\mathbf{x}} = \mathbf{0}$ has no solution. This is the only known chaotic system without fixed points. Moreover, by computing the trace of Jacobi matrix we will learn quickly that the attractor is not loosing it´s energy along the orbit. State space volume is preserved, the rate of contraction $V(t) = V_0 \exp(tr\,\mathbf{J}) = V_0 \exp(z) = V_0$.

For the common set of system parameters, the attractor has the largest Lyapunov exponent 0.014 (weak chaos) and the rest of them 0, -0.014. The so called Kaplan-Yorke dimension of the attractor is $D_{KY} = 3$. The procedure for behavior preserving expansion of dynamical systems into higher dimensions has been described briefly in [3]. In the case of studying system, omitting constant vector **D** the state matrix is antisymmetric and then is generated by angular momentum group. Resulting state trajectory is given by interaction of two limit cycles. Adding four more terms into the system of ordinary differential equations the new four-dimensional dynamical system can be rewriten as

$$\dot{x} = y + yz, \quad \dot{y} = -x + yz, \quad \dot{z} = -xy - y^2 + yw + 1, \quad \dot{w} = -yz \tag{2 }$$

Because of constant term in the third equation the origin is still not an equilibrium point. Semisimple part **R**(**x**) of the system (2) produces invariant dynamics on any hypersphere. Let´s use standard norm on Euklidean space E^4 which leads into the (3a)

$$\frac{d}{dt}\|\mathbf{x}\|^2 = 0, \quad \mathbf{x}^{\mathrm{T}} \cdot \mathbf{R}(\mathbf{x}) \cdot \mathbf{x} < 0 \tag{3a, 3b}$$

Excluding constant term the stability of the origin can be determined by means of (3b).

Topological complexicity of the chaotic attractor can be hardly compared to the intrinsic single or double scroll structures. Upper triangular state matrix builds up the solvable part of the dynamical system. The set $\dot{x} = f(y, z)$, $\dot{y} = g(z)$, $\dot{z} = const$ can be integrated recursively and then it offers an analytic solution. The aim is to establish transformation of coordinates with desired solvable part and remaining last equation $\dot{z} = \omega y$ ($\dot{z} = \omega x$). Then the solution lie on the plane and can be expressed as $z(t) = C_1 \sin \omega t + C_2 \cos \omega t$.

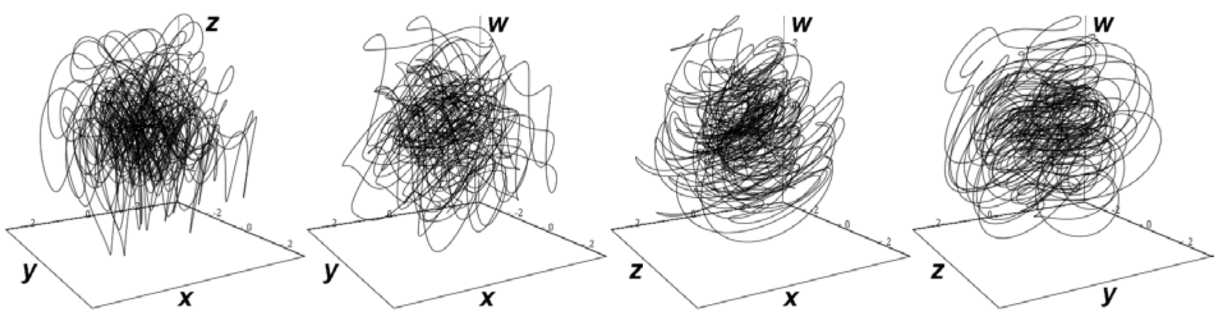

Fig. 1: *Perspective phase projections of proposed 4D chaotic attractor.*

3 CIRCUITRY IMPLEMENTATION

MathCAD 2000 and fourth order Runge-Kutta iteration method has been employed to perform necessary calculations. The numerical analysis of the system (3) is shown on the figure 1. Parameters $t \in \langle 0,400 \rangle$, $\Delta t = 0.2$ have been chosen to ensure sufficient accuracy. Initial conditions were equal (0, 5, 0).

When constructing a chaotic oscillator it is well known that we need to create two or four-terminal nonlinear element. Polynomial nonlinearity can be directly realized by means of analog multiplier. Our circuit works with four quadrant analog multiplier AD633. This device has a transfer function $W = 0.1(X_1 - X_2)(Y_1 - Y_2) + Z$ especially suitable when nonlinear function is given as a product of root factors. Because of internal denominator we usually add noninverting booster as compensation. Although designed circuit is definitely not the simplest (see figure 2), it consists only cheap and commercially available devices. We can partially reduce the circuit elements tangles by using four operational amplifiers (opamp) in a single package TL084. However, when moving the main frequency (about 1 kHz) in the direction to higher values, it is recommended to replace conventional opamp by CFOA (CCII+ followed by voltage buffer) AD844.

Whole circuit is feeded from symmetrical ±15 V sources. Simulated trajectories are demonstrated on the figure 3. Probe oscilloscope has a uniform grid for *x-axis* and *y-axis*, namely *2V/div*. Fourier spectrum of each state variable is continuous and spreaded roughly from zero to 20 kHz.

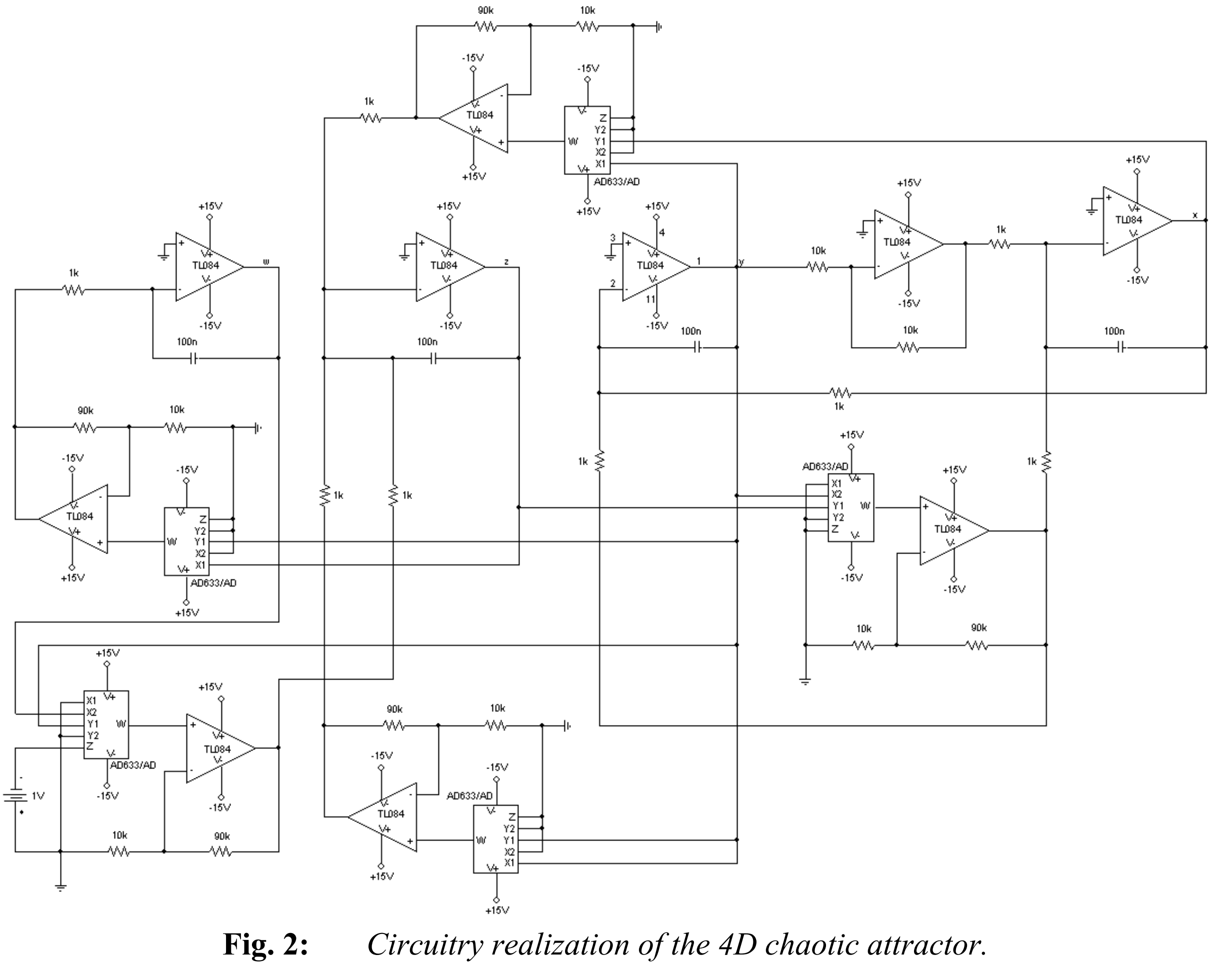

Fig. 2: *Circuitry realization of the 4D chaotic attractor.*

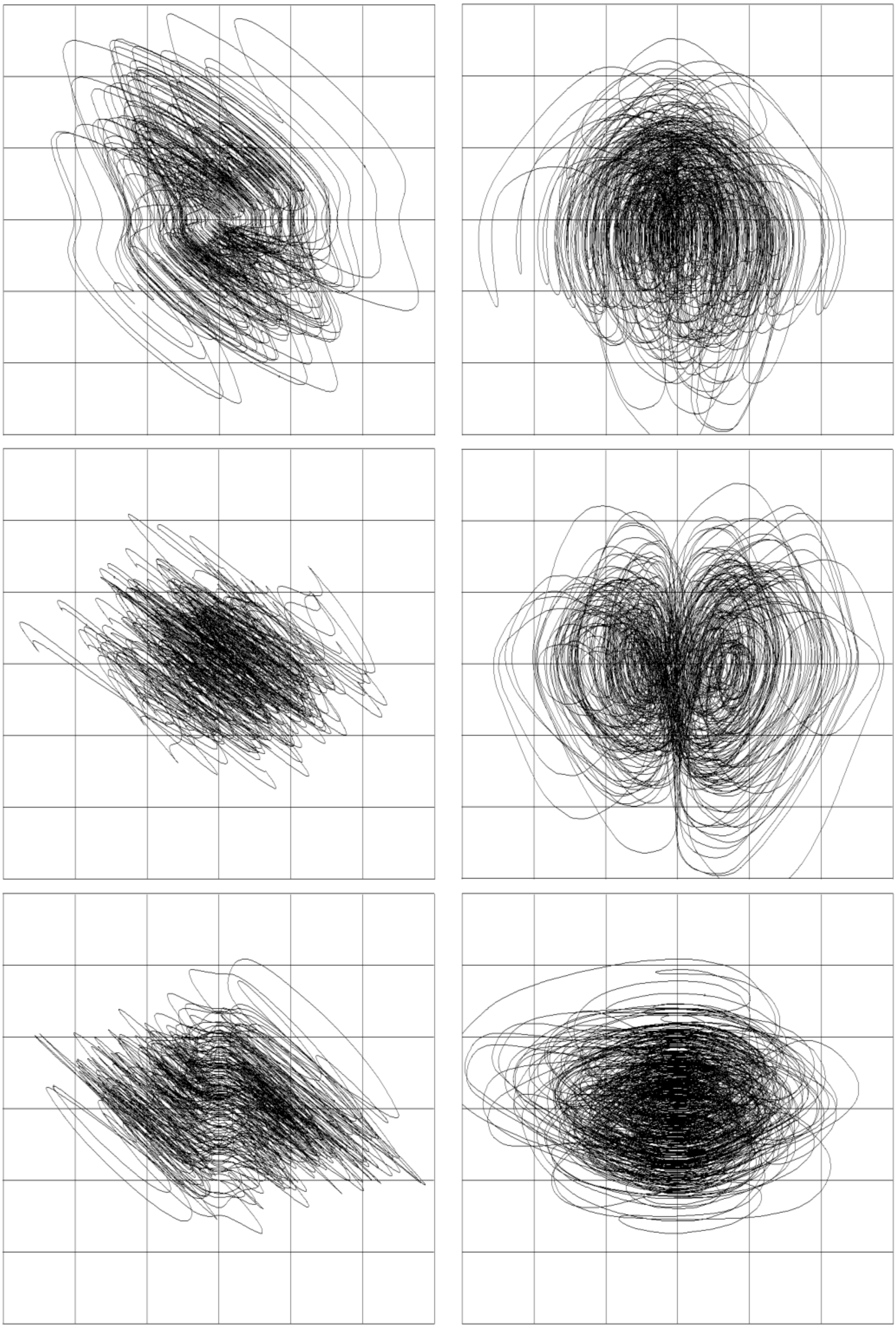

Fig. 3: *Plane projections of the mentioned chaotic attractor.*

4 CONCLUSION

It is evident that the implicitly given 3rd order differential equation can be intuitively realized by a cascade connection of inverting integrators. Basic system (1) can be writeln as

$$\dddot{x} + \dot{x}^3 - \frac{\ddot{x}}{\dot{x}}\left(x + \ddot{x}\right) = 0 \tag{4}$$

This approach is onerous, since the circuit´s feedback will kidnap four multiplier cells, one connected as divider. In addition, it is almost impossible to rewrite (2) into this form.

The new voltage mode chaotic oscillator has been experimentally verified. Required conditions for hyperchaotic solution are assumed to be fulfilled, such as there are at least two positive Lyapunov exponents to obtain the necessary divergency in every 3D projection. Mentioned dynamical system doesn´t need an adjustable parameter, period doubling bifurcation sequence can be traced by introducing various initial condition y_0. Especially parasitic input capacitances of used opamps are the important principles of dynamical behavior. In spite of this, no qualitative changes in state trajectories were observed when more complex circuit models was tested.

To date, the versatility of Levi decomposition has not been generalized to the entire class of chaotic systems with quadratic vector fields. Several stability lemmas including the formulae for finding Lyapunov function have been derived directly from the Lie algebra theory. Two-dimensional systems can be also lifted [5] to 3D by using stereographic mapping. This kind of projection does not guarantee a vector field on the sphere which is well defined at the north pole (the point at ∞). Such systems will play the significant role in future communication techniques. Lately, nonperiodic systems are widely used as broadband signal generators or real-time speech securing.

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