HIGHER-ORDER ALGORITHM IN TIME-DOMAIN FINITE ELEMENT METHOD

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ABSTRACT

The paper deals with the derivation of a higher-order time-domain scheme for Time-Domain Finite Element Method (TD-FEM). An explicit and an implicit time-domain update scheme based on the third order approximation in time are presented.

1 INTRODUCTION

The TD-FEM is based on solving the wave equation [1]

$$\vec{\nabla} \times \left(\frac{1}{\mu_r} \vec{\nabla} \times \vec{E} \right) + \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} = -\mu_0 \frac{\partial \vec{J}}{\partial t}, \tag{1}$$

where $\vec{E}$ denotes an unknown electric field intensity vector, μ_r is relative permeability, μ_0 denotes permeability of vacuum, ε and σ are permittivity and conductivity of media, respectively.

We can use nodal finite elements [1]. Then, the vector equation (1) can be divided into three scalar equations for each component of $\vec{E}$. E.g., the z component is given by

$$-\frac{1}{\mu_r} \vec{\nabla}^2 E_z + \mu_0 \varepsilon \frac{\partial^2 E_z}{\partial t^2} + \mu_0 \sigma \frac{\partial E_z}{\partial t} = -\mu_0 \frac{\partial J_z}{\partial t} \ . \tag{2}$$

The following semi-discrete equation can be obtained by multiplying (2) by the space weighting function N_i, by integrating the product over the finite element, and by applying Green's identity [1]

$$\iiint\limits_V \left\{ \frac{1}{\mu_r} \left(\vec{\nabla} N_i \right) \cdot \left(\vec{\nabla} E_z \right) + \mu_0 \varepsilon N_i \frac{\partial^2 E_z}{\partial t^2} + \mu_0 \sigma N_i \frac{\partial E_z}{\partial t} \right\} dV = -\mu_0 \iiint\limits_V N_i \frac{\partial J_z}{\partial t} dV \ . \tag{3}$$

Now, we have to approximate an unknown electric field using space basis functions N_j

$$E = \sum_{j=1}^{N} u_j N_j \ . \tag{4}$$

Here u_j denotes unknown nodal values of electric field and N is the number of unknown coefficients. Substituting (4) into (3), we can obtain the matrix differential equation [1]

$$\mathbf{T}\frac{d^2\mathbf{u}}{dt^2} + \mathbf{R}\frac{d\mathbf{u}}{dt} + \mathbf{S}\mathbf{u} = -\mathbf{f} \ , \tag{5}$$

where $\mathbf{u}=[u_1, u_2, \ldots, u_N]^T$ denotes the vector of unknown coefficients and **T**, **R**, **S** are square matrices, which terms are given as follows

$$T_{ij} = \mu_0 \varepsilon_0 \iiint_V \varepsilon_r N_i N_j dV \ ,$$

$$R_{ij} = \mu_0 \iiint_V \sigma N_i N_j dV \ , \tag{6}$$

$$S_{ij} = \iiint_V \frac{1}{\mu_r} \left(\vec{\nabla} N_i\right) \cdot \left(\vec{\nabla} N_j\right) dV \ ,$$

In (6), ε_0 and ε_r are permittivity of vacuum and relative permittivity. The vector **f** denotes an excitation vector given by

$$f_i = \mu_0 \iiint_V N_i \frac{\partial J_i}{\partial t} dV \ . \tag{7}$$

2 HIGH-ORDER APPROXIMATION IN TIME DOMAIN

Lagrange polynomial is the most useful approximation for time-domain scheme. The usual approximation in the time domain is based on the second-order Lagrange polynomial [2]. In this paper, the third-order approximation is developed. In the next, the superscript denotes a time-step index. Due to the symmetry, the terms u^{-2}, u^{-1}, u^1 and u^2 denote values related to equidistantly divided time points $3\delta t/2$, $-\delta t/2$, $\delta t/2$, $3\delta t/2$, respectively. We use the third-order general form of Lagrange polynomial given by

$$u(t) = \frac{1}{48(\delta t)^3}\Big[a u^{-2}(2t+\delta t)(2t-\delta t)(2t-3\delta t) + b u^{-1}(2t+3\delta t)(2t-\delta t)(2t-3\delta t) +$$
$$+ c u^{1}(2t+3\delta t)(2t+\delta t)(2t-3\delta t) + d u^{2}(2t+3\delta t)(2t+\delta t)(2t-\delta t)\Big] \ , \tag{8}$$

where a, b, c and d are constants.

Now, we have to compare the derivatives of this polynomial (in given time points $3\delta t/2$, $-\delta t/2$, $\delta t/2$, $3\delta t/2$) with general finite differences [1] in order to obtain constants a, b, c and d. In this case, we get $a = -1$, $b = 3$, $c = -3$, $d = 1$. The polynomial (8) melts into

$$u(t) = \frac{1}{48(\delta t)^3}\Big[-u^{-2}(2t+\delta t)(2t-\delta t)(2t-3\delta t) + 3u^{-1}(2t+3\delta t)(2t-\delta t)(2t-3\delta t) +$$
$$- 3u^{1}(2t+3\delta t)(2t+\delta t)(2t-3\delta t) + u^{2}(2t+3\delta t)(2t+\delta t)(2t-\delta t)\Big]. \tag{9}$$

The first derivative of the polynomial (9) is given by

$$\frac{du(t)}{dt} = \frac{1}{48(dt)^3}\Big[-u^{-2}\left(24t^2 - 24t\delta t - 2\delta t^2\right) + 3u^{-1}\left(24t^2 - 8t\delta t - 18\delta t^2\right) -$$

$$-3u^1\left(24t^2+8t\delta t-18\delta t^2\right)+u^2\left(24t^2+24t\delta t-2\delta t^2\right)\Big] \,. \tag{10}$$

The second derivative of the polynomial (9) can be expressed as

$$\frac{d^2u(t)}{dt^2}=\frac{1}{48(dt)^3}\left[-u^{-2}(48t-24\delta t)+3u^{-1}(48t-8\delta t)-3u^1(48t+8\delta t)+u^2(48t+24\delta t)\right] . \tag{11}$$

Now, we can substitute (9), (10), (11) into the semi-discrete equation (5). In this case, we obtain

$$\begin{gathered}24\cdot\mathbf{T}\cdot\left[-u^{-2}(2t-\delta t)+u^{-1}(6t-\delta t)-u^1(6t+\delta t)+u^2(2t+\delta t)\right]+\\ +2\cdot\mathbf{B}\cdot\left[-u^{-2}\left(12t^2-12t\delta t-\delta t^2\right)+3u^{-1}\left(12t^2-4t\delta t-9\delta t^2\right)-\right.\\ \left.-3u^1\left(12t^2+4t\delta t-9\delta t^2\right)+u^2\left(12t^2+12t\delta t-\delta t^2\right)\right]\\ +\mathbf{S}\left[u^{-2}\left(-8t^3+12t^2\delta t+2t\delta t^2-3\delta t^3\right)+3u^{-1}\left(8t^3-4t^2\delta t-18t\delta t^2+9\delta t^3\right)-\right.\\ \left.-3u^1\left(8t^3+4t^2\delta t-18t\delta t^2-9\delta t^3\right)+u^2\left(8t^3+12t^2\delta t-2t\delta t^2-3\delta t^3\right)\right]+48\delta t^3\mathbf{f} \;.\end{gathered} \tag{12}$$

In order to obtain the time-domain scheme, the equation (12) is multiplied by the function $W(t)$ and integrated in time. This approach is called the weighting of residual in the time domain [2]. Dividing the result by δt, we obtain

$$\begin{gathered}u^{-2}\left[24\mathbf{T}(-2\Theta_1+1)+2\mathbf{B}\delta t(-12\Theta_2+12\Theta_1+1)+\mathbf{S}(\delta t)^2(-8\Theta_3+12\Theta_2+2\Theta_1-3)\right]+\\ +u^{-1}\left[24\mathbf{T}(6\Theta_1-1)+2\mathbf{B}\delta t(36\Theta_2-12\Theta_1-27)+\mathbf{S}(\delta t)^2(24\Theta_3-12\Theta_2-54\Theta_1+27)\right]+\\ +u^{1}\left[24\mathbf{T}(-6\Theta_1-1)+2\mathbf{B}\delta t(-36\Theta_2-12\Theta_1+27)+\mathbf{S}(\delta t)^2(-24\Theta_3-12\Theta_2+54\Theta_1+27)\right]+\\ +u^{2}\left[24\mathbf{T}(2\Theta_1+1)+2\mathbf{B}\delta t(12\Theta_2+12\Theta_1-1)+\mathbf{S}(\delta t)^2(8\Theta_3+12\Theta_2-2\Theta_1-3)\right]+\\ +48(\delta t)^2\mathbf{g} \;,\end{gathered} \tag{13}$$

where coefficients Θ_1, Θ_2, Θ_3 and the vector $\mathbf{g}$ are given as follows

$$\Theta_1=\frac{\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)t\,dt}{\delta t\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)dt}\;,\quad \Theta_2=\frac{\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)t^2dt}{(\delta t)^2\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)dt}\;,\quad \Theta_3=\frac{\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)t^3dt}{(\delta t)^3\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)dt}\;,\quad \mathbf{g}=\frac{\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)\mathbf{f}dt}{\int_{-\frac{3\delta t}{2}}^{\frac{3\delta t}{2}}W(t)dt}\;. \tag{14}$$

Now, we have to set coefficients Θ_1, Θ_2, Θ_3 in order to ensure the stability of the scheme (13). According to the general stability conditions [2], we obtain the following inequalities

$$\Theta_3\geq\frac{7}{4}\Theta_1$$

$$\Theta_3 \le \frac{1}{2}(6\Theta_2 - 1)\Theta_1 \ . \tag{15}$$

We can experimentally show that even in this case, the stability is not ensured for any structure: the stability is the best when choosing Θ_1=0 and Θ_3=0. In this case, the equation (13) melts into

$$u^{-2}\left[\frac{1}{2}\mathbf{T} + \frac{1}{24}(-12\Theta_2 + 1)\delta t\mathbf{B} + \frac{1}{16}(4\Theta_2 - 1)(\delta t)^2\mathbf{S}\right] +$$

$$+u^{-1}\left[-\frac{1}{2}\mathbf{T} + \frac{3}{8}(4\Theta_2 - 3)\delta t\mathbf{B} + \frac{1}{16}(-4\Theta_2 + 9)(\delta t)^2\mathbf{S}\right] +$$

$$+u^{1}\left[-\frac{1}{2}\mathbf{T} + \frac{3}{8}(-4\Theta_2 + 3)\delta t\mathbf{B} + \frac{1}{16}(-4\Theta_2 + 9)(\delta t)^2\mathbf{S}\right] +$$

$$+u^{2}\left[\frac{1}{2}\mathbf{T} + \frac{1}{24}(12\Theta_2 - 1)\delta t\mathbf{B} + \frac{1}{16}(4\Theta_2 - 1)(\delta t)^2\mathbf{S}\right] + (\delta t)^2\mathbf{g} \tag{16}$$

Now, we can extract the general three-step algorithm for the computation of the time response. We have to set $\Theta_2 \ge 3/4$ for the unconditional stability. The minimum dispersion error is reached for Θ_2=3/4. After substituting Θ_2=3/4, transposing equation (16) and re-indexing time steps, we get the implicit algorithm

$$\left[\frac{1}{2}\mathbf{T} - \frac{1}{3}\delta t\mathbf{B} + \frac{1}{8}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n-2} + \left[-\frac{1}{2}\mathbf{T} + \frac{3}{8}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n-1} + \left[-\frac{1}{2}\mathbf{T} + \frac{3}{8}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n} +$$

$$+\left[\frac{1}{2}\mathbf{T} + \frac{1}{3}\delta t\mathbf{B} + \frac{1}{8}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n+1} + (\delta t)^2\mathbf{g} \ . \tag{17}$$

In order to obtain the explicit algorithm, we have to choose Θ_2 so that the multiplicand of **S** in is zero for the time number u^2. This condition is satisfied for Θ_2=1/4. After substituting Θ_2=1/4, transposing equation (16) and re-indexing time steps, we get the explicit algorithm

$$\left[\frac{1}{2}\mathbf{T} - \frac{1}{12}\delta t\mathbf{B}\right]\mathbf{u}^{n-2} + \left[-\frac{1}{2}\mathbf{T} - \frac{3}{4}\delta t\mathbf{B} + \frac{1}{2}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n-1} + \left[-\frac{1}{2}\mathbf{T} + \frac{3}{4}\delta t\mathbf{B} + \frac{1}{2}(\delta t)^2\mathbf{S}\right]\mathbf{u}^{n} +$$

$$+\left[\frac{1}{2}\mathbf{T} + \frac{1}{12}\delta t\mathbf{B}\right]\mathbf{u}^{n+1} + (\delta t)^2\mathbf{g} \ . \tag{18}$$

3 EXAMPLE

The cuboidal resonator with dimensions 150 mm, 180 mm and 130 mm was analyzed. The discretization mesh was set to N = 20 per side of the structure. The problem was solved in the frequency range from 0 to 4 GHz, with 0.5 MHz resolution. The corresponding spectra of the method are not shown here, as they cannot be compared easily. Instead, a list of wave-mode frequencies is generated.

The two-step and three-step algorithms were used for analyzing this resonator. The dispersion errors were found to be the same. On the other hand, the explicit three-step algorithm exhibits better stability for a longer time step.

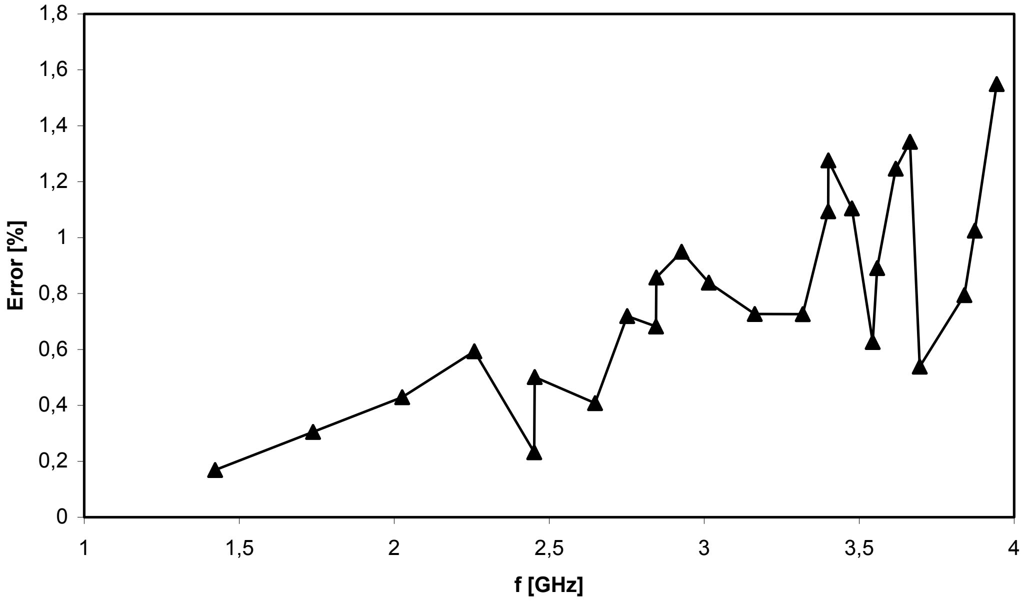

Fig. 1: *Eigenfrequency error for TM modes, N=20*

4 CONCLUSION

The explicit scheme based on the three-step algorithm (18) exhibits better stability than the explicit scheme based on the two-step algorithm presented in [2], because the explicit two-step algorithm is set at Θ_2=0 and accordingly Dirac pulse is used as a weighting function in the time domain. The explicit three-step algorithm is set at Θ_2=1/4 and accordingly constant function is used as a weighting function in the time domain.

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